Formation Control of Mobile Robots

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FORMATION CONTROL OF MOBILE ROBOTS

– Monograph –

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Preface

Technological advances in communication systems and the growing ease in making small, low power and inexpensive mobile machines make it possible to deploy a group of networked mobile vehicles to offer potential advantages in performance, redundancy, fault tolerance, and robustness exceeding the abilities of single machines. Formation is an extremely useful tool mimicking from biological systems to man-made teams of vehicles, mobile sensors and embedded robotic systems to perform tasks such as jointly moving in a synchronized manner or deploying over a given region with applications to search, rescue, coverage, surveillance, reconnaissance and cooperative transportation. Inspired by the progress in the field, I present this monograph, *Formation Control of Mobile Robots*, to postgraduate students, researchers and engineers with control background in Mechanical Engineering, Electrical Engineering and Applied Mathematics. The book mainly consists of my research over the last 7 years in the area of control of single nonholonomic vehicles and formation control of multiple agents and nonholonomic vehicles. Specifically, I focus on unicycle-type mobile robots. The book consists of seven chapters and one appendix.

Chapter 1 classifies basic motion control tasks for nonholonomic wheeled mobile robots of unicycle type. Their modeling and main control properties on the plane are then provided. This chapter sets out the basic material for the subsequent chapters.

Chapter 2 presents time-varying global adaptive controllers at the torque level that simultaneously solve both tracking and stabilization for mobile robots. Both full state feedback and output feedback are considered. Then a constructive controller is presented to solve a path following problem. The controller synthesis is based on several special coordinate transformations, Lyapunov's direct method and the backstepping technique.

Chapter 3 deals with a constructive method to design cooperative controllers that force a group of N mobile agents to achieve a particular formation in terms of shape and orientation while avoiding collisions between themselves. The control development is based on new local potential functions, which attain the minimum value when the desired formation is achieved, and are equal to infinity when a collision occurs. The proposed controller development is then extended to formation control of nonholonomic unicycle-type mobile robots.

Chapter 4 investigates formation control of a group of unicycle-type mobile robots with a little amount of inter-robot communication. A combination of the virtual structure and path-tracking approaches is used to derive the formation architecture. For each robot, a coordinate transformation is first derived to cancel the velocity quadratic terms. An observer is then designed to globally exponentially/asymptotically estimate the unmeasured velocities. An output feedback controller is designed for each robot in such a way that the derivative of the path parameter is left as a free input to synchronize the robots' motion.

In Chapter 5, a constructive method, which is the base for the next two chapters, is presented to design bounded cooperative controllers that force a group of N mobile agents with limited sensing ranges to stabilize at a desired location, and guarantee no collisions between the agents. The dynamics of each agent is described by a single integrator. The control development is based on new general potential functions, which attain the minimum value when the desired formation is achieved, and are equal to infinity when a collision occurs. A *p times differential bump function* is embedded into the potential functions to deal with the agent limited sensing ranges. An alternative to Barbalat's lemma is used to analyze stability of the closed loop system. The proposed formation stabilization solution is then extended to solve a formation tracking problem.

In Chapter 6, based on the material presented in Chapter 5, a constructive method is presented to design cooperative controllers that force a group of N unicycle-type mobile robots with limited sensing ranges to perform desired formation tracking, and guarantee no collisions between the robots. Each robot requires only measurement of position and velocity of itself, and those of the robots within its sensing range for feedfack. Physical dimensions and dynamics of the robots are also considered in the control design. Smooth and p times differential bump functions are incorporated into novel potential functions to design a formation tracking control system. Despite the robot limited sensing ranges, no switchings are needed to solve the collision avoidance problem.

Chapter 7, based on the material presented in Chapters 5 and 6, presents a constructive method to design output-feedback cooperative controllers that force a group of N unicycle-type mobile robots with limited sensing ranges to perform desired formation tracking, and guarantee no collisions between the robots. The robot velocities are not required for control implementation. For each robot an interlaced observer, which is a reduced order observer plus an interlaced term, is designed to estimate the robot unmeasured velocities. The interlaced term is determined after the formation control design is completed to avoid difficulties due to observer errors and consideration of collision avoidance. Smooth and p times differentiable bump functions are incorporated into novel potential functions to design a formation tracking control system.

The Appendix A provides the reader with the mathematical background utilized in the control design and stability analysis such as Lyapunov stability theory, a series of Barbalat like lemmas, and p-times differentiable and smooth bump functions.

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Contents

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Modeling and Control Properties of Single Mobile Robots

In this chapter, basic motion control tasks for nonholonomic wheeled mobile robots of unicycle type will be first classified. Their modeling and main control properties on the plane will be then provided. This chapter sets out the basic material that will be used in the subsequent chapters.

1.1 Introduction

In automatic control, feedback improves system performance by allowing the successful completion of a task even in the presence of external disturbances and initial errors, and inaccuracy of the system parameters. To this end, real-time sensor measurements are used to reconstruct the robot state. Throughout this study, the latter is assumed to be available at every instant, as provided by proprioceptive (e.g., odometry) or exteroceptive (sonar, laser) sensors. In some cases, we also assume that the robot velocities are measurable or constructible from position measurements.

We will limit our analysis to the case of a robot workspace free of obstacles. In fact, we implicitly consider the robot controller to be embedded in a hierarchical architecture in which a higher-level planner solves the obstacle avoidance problem and provides a series of motion goals to the lower control layer. In this perspective, the controller deals with the basic issue of converting ideal plans into actual motion execution. The specific robotic system considered is a vehicle whose kinematic model approximates the mobility of a three wheeled car. The configuration of this robot is represented by the position and orientation of its main body in the plane. Two velocity inputs are available for motion control. This situation covers in a realistic way many of the existing robotic vehicles. Moreover, the three wheel robot is the simplest nonholonomic vehicle that displays the general characteristics and the difficult maneuverability of higher dimensional systems, e.g., of a four wheel car or a car towing trailers. As a matter of fact, the control results presented here can be directly extended to more general kinematics, namely to all mobile robots admitting a chained-form representation.

The nonholonomic nature of the three wheel car-like robot is related to the assumption that the robot wheels roll without slipping. This implies the presence of a nonintegrable set of first-order differential constraints on the configuration variables. While these nonholonomic constraints reduce the instantaneous motions that the robot can perform, they still allow global controllability in the configuration space. This unique feature leads to some challenging problems in the synthesis of feedback controllers, which parallel the new research issues arising in nonholonomic motion planning. Indeed, the wheeled mobile robot application has triggered the search for innovative types of feedback controllers that can be used also for more general nonlinear systems that describe motion of more complicated vehicles such as ocean and air vehicles.

1.2 Basic motion tasks

In order to derive the most suitable feedback controllers for each case, it is convenient to classify the possible motion tasks as follows:

- Point-to-point motion: The robot must reach a desired goal configuration starting from a given initial configuration, see Figure 1.1A.
- Path following: The robot must reach and follow a geometric path in the cartesian space starting from a given initial configuration (on or off the path), see Figure 1.1B.
- Trajectory tracking: The robot must reach and follow a trajectory in the cartesian space (i.e., a geometric path with an associated timing law) starting from a given initial configuration (on or off the trajectory), see Figure 1.1C.

The three tasks are sketched in Figure 1.1, with reference to a three wheel car-like robot. Execution of these tasks can be achieved using either feedforward commands, or feedback control, or a combination of the two. Indeed, feedback solutions exhibit an intrinsic degree of robustness.

Using a more control-oriented terminology, the point-to-point motion task is a stabilization problem for an (equilibrium) point in the robot state space. When using a feedback strategy, the point-to-point motion task leads to a state regulation control problem for a point in the robot state space. Posture stabilization is another frequently used term. Without loss of generality, the goal can be taken as the origin of the n-dimensional robot configuration space. Contrary to the usual situation, tracking and path following are easier than regulation for a nonholonomic wheeled mobile robots. An intuitive explanation of this can be given in terms of a comparison between the number

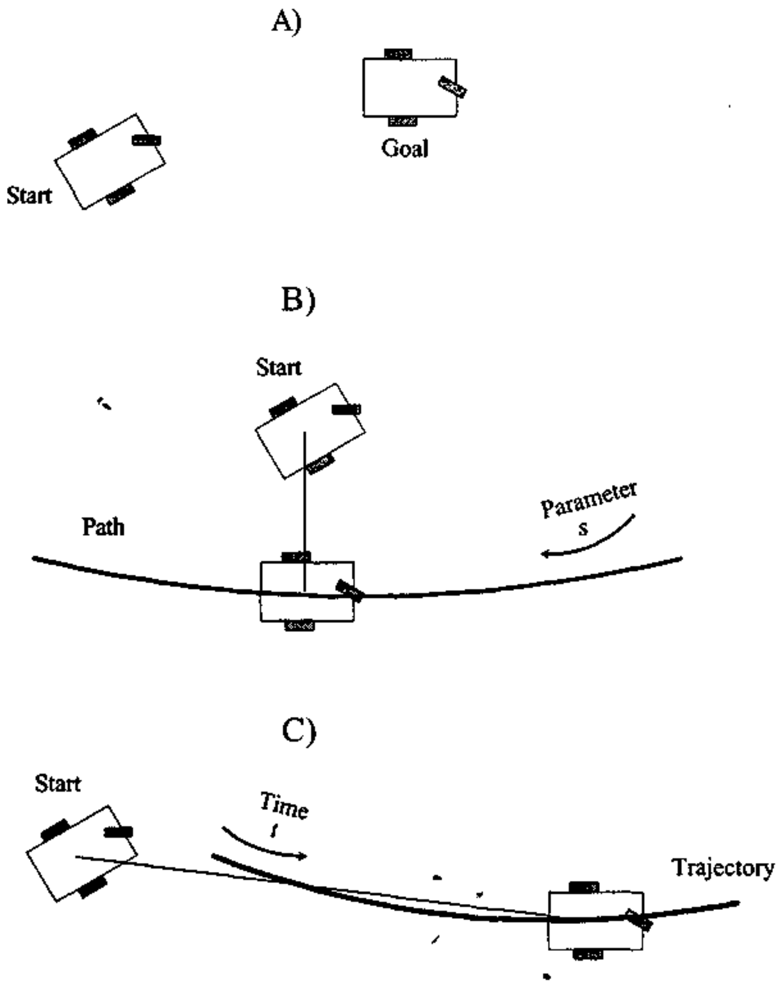


Fig. 1.1. Robot parameters.

of controlled variables (outputs) and the number of control inputs. For the unicycle-like vehicle or three wheel car-like robot, two input commands are available while three variables (position and orientation) are needed to determine its configuration. Thus, regulation of the wheeled mobile robot posture to a desired configuration implies zeroing three independent configuration errors. When tracking a trajectory, and following a path, instead, the output has the same dimension as the input and the control problem is square.

In the path following task, the controller is given a geometric description of the assigned cartesian path. This information is usually available in a parameterized form expressing the desired motion in terms of a path parameter, which may be in particular the arc length along the path. For this task, time dependence is not relevant because one is concerned only with the geometric displacement between the robot and the path. In this context, the time evolution of the path parameter is usually free and, accordingly, the command inputs can be arbitrarily scaled with respect to time without changing the resulting robot path. It is then customary to set the robot forward velocity (one of the two inputs) to an arbitrary constant or time-varying value, leaving the second input available for control. The path following problem is thus rephrased as the stabilization to zero of a suitable scalar path error function using only one control input.

In the trajectory tracking task, the robot must follow the desired cartesian path with a specified timing law (equivalently, it must track a moving reference robot). Although the trajectory can be split into a parameterized geometric path and a timing law for the parameter, such separation is not strictly necessary. Often, it is simpler to specify the workspace trajectory as the desired time evolution for the position of some representative point of the robot. The trajectory tracking problem consists then in the stabilization to zero of the two-dimensional cartesian error using both control inputs.

The point stabilization problem can be formulated in a local or in a global sense, the latter meaning that we allow for initial configurations that are arbitrarily far from the destination. The same is true also for path following and trajectory tracking, although locality has two different meanings in these tasks. For path following, a local solution means that the controller works properly provided we start sufficiently close to the path; for trajectory tracking, closeness should be evaluated with respect to the current position of the reference robot. The amount of information that should be provided by a high-level motion planner varies for each control task. In point-to-point motion, information is reduced to a minimum (i.e., the goal configuration only) when a globally stabilizing feedback control solution is available. However, if the initial error is large, such a control may produce erratic behavior and/or large control effort, which are unacceptable in practice. On the other hand, a local feedback solution requires the definition of intermediate subgoals at the task planning level in order to get closer to the final desired configuration. For the other two motion tasks, the planner should provide a path which is kinematically feasible (namely, which complies with the nonholonomic constraints of the specific vehicle), so as to allow its perfect execution in nominal conditions. While for an omnidirectional robot any path is feasible, some degree of geometric smoothness is in general required for nonholonomic robots. Nevertheless, the intrinsic feedback structure of the driving commands enables to recover transient errors due to isolated path discontinuities. Note also that the unfeasibility arising from a lack of continuity in some higher-order derivative of the path may be overcome by appropriate motion timing. For example, paths with discontinuous curvature (like the Reeds and Shepp optimal paths under maximum curvature constraint) can be executed by the real axle midpoint of a three wheel car-like vehicle provided that the robot is allowed to stop, whereas paths with discontinuous tangent are not feasible. In this analysis, the selection of the robot representative point for path/trajectory planning is critical. The timing profile is the additional item needed in trajectory tracking control tasks. This information is seldom provided by current motion planners, also because the actual dynamics of the specific robot are typically neglected at this level. The above example suggests that it may be reasonable to enforce already at the planning stage requirements such as "move slower where the path curvature is higher".

1.3 Modeling and control properties

1.3.1 Modeling

Through out this book, we consider the unicycle-type mobile robot, which under an assumption of no wheel slips has the following dynamics [1]:

$$\begin{aligned} \dot{\eta} &= J(\eta)\omega, \\ M\dot{\omega} &= -C(\dot{\eta})\omega - D\omega + \tau \cdot \end{aligned} \tag{1.1}$$

where $\eta = [x \ y \ \phi]^T$ denotes the position (x, y), the coordinates of the middle point, P_0, between the left and right driving wheels, and heading ϕ of the robot coordinated in the earth-fixed frame OXY, see Figure 1.2, $\omega = [\omega_1 \ \omega_2]^T$ with ω_1 and ω_2 being the angular velocities of the wheels of the robot, $\tau = [\tau_1 \ \tau_2]^T$ with τ_1 and τ_2 being the control torques applied to the wheels of the robot. The rotation matrix $J(\eta)$, mass matrix M, Coriolis matrix $C(\dot{\eta})$, and damping matrix D in (1.1) are given by

$$\begin{aligned} J(\eta) &= \frac{r}{2}\begin{bmatrix} \cos(\phi) & \cos(\phi) \\ \sin(\phi) & \sin(\phi) \\ \frac{1}{b} & -\frac{1}{b} \end{bmatrix}, \quad M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{11} \end{bmatrix}, \\ C(\dot{\eta}) &= \begin{bmatrix} 0 & c\dot{\phi} \\ -c\dot{\phi} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \end{aligned} \tag{1.2}$$

with

$$\begin{aligned} c &= \frac{1}{2b}r^2 m_c a, \ m_{11} = \frac{1}{4b^2}r^2(mb^2 + I) + I_w, \ m_{12} = \frac{1}{4b^2}r^2(mb^2 - I), \\ m &= m_c + 2m_w, \ I = m_c a^2 + 2m_w b^2 + I_c + 2I_m \end{aligned} \tag{1.3}$$

where m_c and m_w are the masses of the body and wheel with a motor; I_c, I_w and I_m are the moments of inertia of the body about the vertical axis through P_c (center of mass), the wheel with the rotor of a motor about the wheel axis, and the wheel with the rotor of a motor about the wheel diameter, respectively; r, a and b are defined in Figure 1.2; the nonnegative constants d_{11} and d_{22} are the damping coefficients. If these damping coefficients are zero, we have an undamped case. On the other hand, if the damping coefficients are positive, we have a damped case. Through out the book, we take the physical parameters from [1]: $b = 0.75$, $a = 0.3$, $r = 0.15$, $m_c = 30$, $m_w = 1$, $I_c = 15.625$, $I_w = 0.005$, $I_m = 0.0025$, $d_{11} = d_{22} = 5$ with appropriate units for numerical simulations

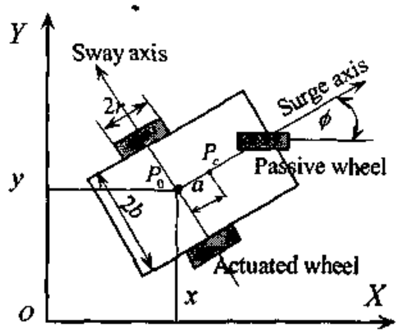


Fig. 1.2. Robot parameters.

For convenience, we convert the wheel angular velocities (ω_1, ω_2) of the robot to its linear, v, and angular, w, velocities by:

$$\varpi = B^{-1}\omega,\ B = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} \tag{1.4}$$

where $\varpi = [v\ \ w]^T$, and B is invertible since $\det(B) = -2b/r$. With (1.4), we can write the robot dynamics (1.1) as follows:

$$\begin{aligned} \dot{x} &= v\cos(\phi) \\ \dot{y} &= v\sin(\phi) \\ \dot{\phi} &= w \\ \overline{M}\dot{\varpi} &= -\overline{C}(w)\varpi - \overline{D}\varpi + \overline{B}\tau \end{aligned} \tag{1.5}$$

where

$$\overline{M} = B^{-1}MB = \begin{bmatrix} \bar{m}_{11i} & 0 \\ 0 & \bar{m}_{22i} \end{bmatrix},$$
$$\overline{C}(w) = B^{-1}C(\dot{\eta})B = \begin{bmatrix} 0 & -bcw \\ \frac{c}{b}w & 0 \end{bmatrix},$$
$$\overline{D} = B^{-1}DB = \begin{bmatrix} \bar{d}_{11} & \bar{d}_{12} \\ \bar{d}_{21} & \bar{d}_{22} \end{bmatrix}, \ \overline{B} = B^{-1},$$
$$\bar{m}_{11} = m_{11} + m_{12}, \bar{m}_{22} = m_{11} - m_{12},$$
$$\bar{d}_{11} = \frac{1}{2}(d_{11i} + d_{22}), \bar{d}_{12} = \frac{b}{2}(d_{11} - d_{22}),$$
$$\bar{d}_{21} = \frac{1}{2b}(d_{11} - d_{22}), \bar{d}_{22} = \frac{1}{2}(d_{11} + d_{22}). \tag{1.6}$$

1.3.2 Control properties

Since the last equation of (1.5) is a square system if we consider the robot velocities v and w as outputs and the torques τ_1 and τ_2 as inputs, we only need to investigate control properties of the first three equations of (1.5), i.e. we investigate control properties of the robot kinematic model:

$$\begin{aligned} \dot{x} &= v\cos(\phi) \\ \dot{y} &= v\sin(\phi) \\ \dot{\phi} &= w \end{aligned} \tag{1.7}$$

From the first two equations of (1.7), the nonholonomic constraint is

$$\dot{x}\sin(\phi) - \dot{y}\cos(\phi) = 0. \tag{1.8}$$

Controllability at a point

The tangent linearization of (1.7) at any point η_e is the linear system

$$\dot{\tilde{\eta}} = \begin{bmatrix} \cos(\phi_e) \\ \sin(\phi_e) \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w, \quad \tilde{\eta} = \eta - \eta_e \tag{1.9}$$

that is clearly not controllable. This implies that a linear controller will never achieve posture stabilization, not even in a local sense. In order to study the controllability of the unicycle, we need therefore to use tools from nonlinear control theory [2]. Let's define

$$g_1 = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.10}$$

It is easy to check that the accessibility rank condition is satisfied globally (at any η_e), since

$$\text{rank}\,[g_1 \quad g_2 \quad [g_1, g_2]] = 3 \tag{1.11}$$

being the Lie bracket $[g_1, g_2]$ of the two input vector fields

$$[g_1, g_2] = \frac{\partial g_2}{\partial \eta} g_1 - \frac{\partial g_1}{\partial \eta} = \begin{bmatrix} \sin(\phi) \\ -\cos(\phi) \\ 0 \end{bmatrix}. \tag{1.12}$$

Since the system is driftless, condition (1.11) implies its controllability. Controllability can also be shown constructively, i.e., by providing an explicit sequence of maneuvers bringing the robot from any start configuration (x_s, y_s, ϕ_s) to any desired goal configuration (x_g, y_g, ϕ_g). Since the unicycle can rotate on itself, this task is simply achieved by an initial rotation on (x_s, y_s) until the unicycle is oriented toward (x_g, y_g), followed by a translation to the goal position, and by a final rotation on (x_g, y_g) so as to align ϕ with ϕ_g. As for the stabilizability of system(1.9) to a point, the failure of the previous linear analysis indicates that exponential stability cannot be achieved by smooth feedback [3]. Things turn out to be even worse: if smooth (in fact, even continuous) time-invariant feedback laws are used, Lyapunov stability is out of reach. This negative result is established on the basis of a necessary condition due to Brockett [4]: smooth stabilizability of a driftless regular system (i.e., such that the input vector fields are well defined and linearly independent at η_e) requires a number of inputs equal to the number of states. The above obstruction has a deep impact on the control design. In fact, to obtain a posture stabilizing controller it is either necessary to give up the continuity requirement and/or to resort to time-varying control laws.

Controllability about a trajectory

Given a desired cartesian motion for the unicycle, it may be convenient to generate a corresponding state trajectory $\eta_d(t) = (x_d(t), y_d(t), \phi_d(t))$. In order to be feasible, the latter must satisfy the nonholonomic constraint on the vehicle motion or, equivalently, be consistent with the equation (1.9). The generation of $q_d(t)$ and of the corresponding reference velocity inputs $v_d(t)$ and $w_d(t)$ will be addressed properly.

Defining the state tracking error as $\tilde{\eta} = \eta - \eta_d$ and the input variations as $\tilde{v} = v - v_d$ and $\tilde{w} = w - w_d$, the tangent linearization of system (1.9) about the reference trajectory is

$$\dot{\tilde{\eta}} = \begin{bmatrix} 0 & 0 & -v_d \sin(\phi_d) \\ 0 & 0 & v_d \cos(\phi_d) \\ 0 & 0 & 0 \end{bmatrix} \tilde{\eta} + \begin{bmatrix} \cos(\phi_d) & 0 \\ \sin(\phi_d) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} = A(t)\tilde{\eta} + B(t) \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}. \tag{1.13}$$

Since the linearized system is time-varying, a necessary and sufficient controllability condition is that the controllability Gramian is nonsingular. However, a simpler analysis can be conducted by defining the state tracking error through a rotation matrix as

$$\tilde{\eta}_R = \begin{bmatrix} \cos(\phi_d) & \sin(\phi_d) & 0 \\ -\sin(\phi_d) & \cos(\phi_d) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\eta}. \tag{1.14}$$

Using (1.9), we obtain

$$\dot{\tilde{\eta}}_R = \begin{bmatrix} 0 & w_d & 0 \\ -w_d & 0 & v_d \\ 0 & 0 & 0 \end{bmatrix} \tilde{\eta}_R + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}. \tag{1.15}$$

When v_d and w_d are constant, the above linear system becomes time-invariant and controllable, since matrix

$$C = [B \quad AB \quad A^2B] = \begin{bmatrix} 1 & 0 & 0 & 0 & -w_d^2 & v_d w_d \\ 0 & 0 & -w_d & v_d & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.16}$$

has rank 3 provided that either v_d or w_d are nonzero. Therefore, we conclude that the kinematic system (1.9) can be locally stabilized by linear feedback about trajectories which consist of linear or circular paths, executed with constant velocity.

Feedback linearizability

Based on the previous discussion, it is easy to see that the driftless nonholonomic system (1.9) cannot be transformed into a linear controllable one using static state feedback. In particular, the controllability condition (1.11) implies that the distribution generated by vector fields g_1 and g_2 is not involutive, thus violating the necessary condition for full state feedback linearizability [2]. However, when matrix

$$G(\eta) = \begin{bmatrix} \cos(\phi) & 0 \\ \sin(\phi) & 0 \\ 0 & 1 \end{bmatrix} \tag{1.17}$$

has full column rank, m equations can always be transformed via feedback into simple integrators (input-output linearization and decoupling). The choice of the linearizing outputs is not unique and can be accommodated for special purposes. An interesting example is the following. Define the two outputs as

$$
\begin{aligned}
y_1 &= x + d\cos(\phi) \\
y_2 &= y + d\sin(\phi)
\end{aligned}
\tag{1.18}
$$

with $d \neq 0$, i.e., the cartesian coordinates of a point displaced at a distance d along the main axis of the unicycle.

Using the globally defined state feedback

$$
\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi)/d & \cos(\phi)/d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\tag{1.19}
$$

the unicycle kinematic is equivalent to

$$
\begin{aligned}
\dot{y}_1 &= u_1 \\
\dot{y}_1 &= u_1 \\
\dot{\phi} &= \frac{u_2\cos(\phi) - u_1\sin(\phi)}{d}.
\end{aligned}
\tag{1.20}
$$

As a consequence, a linear feedback controller for $u = (u_1, u_2)$ will make the point B track any reference trajectory, even with discontinuous tangent to the path (e.g., a square without stopping at corners). Moreover, it is easy to show that the internal state evolution $\phi(t)$ is bounded. This approach, however, will not be pursued in this book because of its limited interest since the robot orientation $\phi(t)$ is not controlled.

Chained forms

The existence of canonical forms for kinematic models of nonholonomic robots allows a general and systematic development of both open-loop and closedloop control strategies. The most useful canonical structure is the chained form, which in the case of two-input systems is

$$
\begin{aligned}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_2 u_1 \\
&\vdots \\
\dot{z}_n &= z_{n-1} u_1.
\end{aligned}
\tag{1.21}
$$

It has been shown that a two-input driftless nonholonomic system with up to $n = 4$ generalized coordinates can always be transformed in chained form by static feedback transformation [5]. As a matter of fact, most (but not all) wheeled mobile robots can be transformed in chained form. For the kinematic model (1.9) of the unicycle, we introduce the following globally defined coordinate transformation

$$
\begin{aligned}
z_1 &= \phi \\
z_2 &= x\cos(\phi) + y\sin(\phi) \\
z_3 &= x\sin(\phi) - y\cos(\phi)
\end{aligned}
\tag{1.22}
$$

and a static state feedback

$$
\begin{aligned}
v &= u_2 + z_3 u_1 \\
w &= u_1
\end{aligned}
\tag{1.23}
$$

to give

$$
\begin{aligned}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_2 u_1 .
\end{aligned}
\tag{1.24}
$$

Note that (z_2, z_3) is the position of the unicycle in a rotating left-handed frame having the z_2 axis aligned with the vehicle orientation. Equation (1.24) is another example of static input-output linearization, with z_1 and z_2 as linearizing outputs. We note also that the transformation in chained form is not unique.

1.4 Notes and references

This chapter sets out the material about the basic motion tasks, mathematical model and control properties of the unicycle-type mobile robots that will be used in the subsequent chapters. Mathematical model and control properties of other types of mobile robots are given in [6] and [7], and of ocean vehicles (robots) are given in [8] and [9]. It was also pointed out that regulation/stabilization is much more difficult than tracking and path following for a nonholonomic wheeled mobile robots. Since the rest of the book will not address the control problem for the chain form (1.21) or (1.24), the reader is referred to [10] for details on controlling a class of nonholonomic systems with strong nonlinear drifts. This class covers the chain form (1.21).

2

Control of Single Mobile Robots

This chapter first presents time-varying global adaptive controllers at the torque level that simultaneously solve both tracking and stabilization for mobile robots. Both full state feedback and output feedback are considered. Then a constructive controller is presented to solve the path following problem. The controller synthesis is based on several special coordinate transformations, Lyapunov's direct method and the backstepping technique briefly given in Appendix A, Sections A.1 and A.4.

2.1 Simultaneous tracking and stabilization: Full state feedback

2.1.1 Problem statement

We consider a mobile robot with two actuated wheels in Chapter 1. For convenience, we rewrite the equations of motion here:

$$\begin{aligned} \dot{\eta} &= J(\eta)\omega \\ M\dot{\omega} &= -C(\dot{\eta})\omega - D\omega + \tau \end{aligned} \tag{2.1}$$

where all the state variables and parameters are defined in Section 1.3.1, Chapter 1. We assume that the reference trajectory is generated by the virtual robot:

$$\begin{aligned} \dot{x}_d &= \cos(\phi_d)u_{1d} \\ \dot{y}_d &= \sin(\phi_d)u_{1d} \\ \dot{\phi}_d &= u_{2d} \end{aligned} \tag{2.2}$$

where (x_d, y_d, ϕ_d) are the position and orientation of the virtual robot; u_{1d} and u_{2d} are the linear and angular velocities of the virtual robot, respectively.

Control objective: Under Assumptions 2.1 and 2.2, design the control τ to force the position and orientation, (x, y, ϕ) of the real robot (2.1) to globally asymptotically track (x_d, y_d, ϕ_d) generated by (2.2).

Assumption 2.1 *The reference signals u_{1d}, $\dot{u}_{1d}$, $\ddot{u}_{1d}$, u_{2d} and $\dot{u}_{2d}$ are bounded. In addition, one of the following conditions holds:*

$$C1. \int_0^\infty \left(|u_{1d}(t)| + |u_{2d}(t)| + |\dot{u}_{1d}(t)|\right) dt \leq \mu_1,$$

$$C2. \int_0^\infty \left(|u_{1d}(t)| + |\dot{u}_{1d}(t)|\right) dt \leq \mu_{21} \text{ and } |u_{2d}(t)| \geq \mu_{22}, \ \forall\ 0 \leq t < \infty,$$

$$C3. |u_{1d}(t)| \geq \mu_3, \quad \forall\ 0 \leq t < \infty$$

where μ_1 and μ_{21} are nonnegative constants, μ_{22} and μ_3 are strictly positive constants.

Assumption 2.2 *All of the robot parameters are constants but unknown, and lie in a compact set.*

Remark 2.3. The problem of set-point regulation/stabilization, tracking a path approaching a set-point is included in C1. Tracking linear and circular paths belongs to C3. Condition C2 implies that the case, where the robot linear velocity is zero or approaches zero and its angular velocity is of sinusoidal type, is excluded. The reason is that our control approach is to introduce a sinusoid signal in the robot angular velocity virtual control to handle set-point stabilization/regulation. Therefore, this case is excluded to avoid two signals from canceling each other. If the reference velocity u_{2d} is known completely in advance, the above case can be included.

Remark 2.4. The problem of simultaneous stabilization and tracking not only is of theoretical interest but also possesses some advantages over the use of separate stabilization and tracking controllers such as only one controller and transient improvement because of no switching. Moreover, if the switching time is unknown, a separate stabilization and tracking control approach cannot be used.

2.1.2 Control design

We interpret the tracking errors as

$$\begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_d \\ y - y_d \\ \phi - \phi_d \end{bmatrix}. \tag{2.3}$$

Indeed convergence of (x_e, y_e, ϕ_e) to zero implies that of $(x-x_d, y-y_d, \phi-\phi_d)$. Using (2.3), (2.2) and the kinematic part of (2.1), we have the kinematic tracking errors:

$$\begin{aligned}\dot{x}_e &= ru_1 - u_{1d}\cos(\phi_e) + rb^{-1}y_e u_2\\ \dot{y}_e &= u_{1d}\sin(\phi_e) - rb^{-1}x_e u_2\\ \dot{\phi}_e &= rb^{-1}u_2 - u_{2d}\end{aligned} \tag{2.4}$$

where $u_1 = 0.5(\omega_1 + \omega_2)$, $u_2 = 0.5(\omega_1 - \omega_2)$. Now if u_1 and u_2 are considered as virtual controls, we can see directly from (2.4) that x_e and ϕ_e can be stabilized by u_1 and u_2. Motivated by the car driving practice, we will use ϕ_e as a virtual control to stabilize y_e. Toward this end, we introduce the coordinate transformation:

$$\begin{aligned}z_e &= \phi_e + \arcsin\left(\frac{k(t)y_e}{\Omega_1}\right),\\ k(t) &= \lambda_1 u_{1d} + \lambda_2\cos(\lambda_3 t),\\ \Omega_1 &= \sqrt{1 + x_e^2 + y_e^2}\end{aligned} \tag{2.5}$$

where λ_i, $i = 1, 2, 3$ are positive constants such that $|k(t)| < 1$, $\forall t$. They will be specified later. It is seen that (2.5) is well defined. We now use (2.5) to write the tracking error dynamics as:

$$\begin{aligned}\dot{x}_e &= ru_1 + f_x + b^{-1}y_e u_2,\\ \dot{y}_e &= -ku_{1d}\Omega_1^{-1}y_e + f_y - rb^{-1}x_e u_2,\\ \dot{z}_e &= rb^{-1}u_2(1 - k\Omega_1^{-1}x_e) + f_z + rg_z u_1,\\ M\dot{\omega} &= -C(\dot{\eta})\omega - D\omega + \tau\end{aligned} \tag{2.6}$$

where for simplicity of presentation, we have dropped the time argument of $k(t)$, and have defined:

$$\begin{aligned}\Omega_2 &= \sqrt{1 + x_e^2 + (1 - k^2)y_e^2},\\ f_x &= -u_{1d}(\cos(z_e)\Omega_1^{-1}\Omega_2 + \sin(z_e)k\Omega_1^{-1}y_e),\\ f_y &= -u_{1d}((\cos(z_e) - 1)\Omega_1^{-1}\Omega_2 + \sin(z_e)k\Omega_1^{-1}y_e),\\ f_z &= -u_{2d} + \Omega_2^{-1}(\dot{k}y_e + kf_y - k\Omega_1^{-2}(x_e f_x + y_e f_y)),\\ g_z &= -k\Omega_1^{-2}\Omega_2^{-1}x_e y_e.\end{aligned} \tag{2.7}$$

The effort, we have made so far, is to have the term $-ku_{1d}\Omega_1^{-1}y_e$ in the y_e-dynamics, and to put the tracking error dynamics in a triangular form of (2.6). We now design τ to stabilize (2.6) in two steps.

Step 1. Define the virtual velocity tracking errors $\tilde{u}_1$ and $\tilde{u}_2$ as

$$\tilde{u}_1 = u_1 - u_{1c}, \quad \tilde{u}_2 = u_2 - u_{2c} \tag{2.8}$$

where u_{1c} and u_{2c} are the virtual controls of u_1 and u_2. To design u_{1c}, we take the Lyapunov function:

$$V_1 = r^{-1}(\Omega_1 - 1) + 0.5\gamma_{11}^{-1}\tilde{\theta}_{11}^2 \tag{2.9}$$

where γ_{11} is a positive constant, $\tilde{\theta}_{11} = \theta_{11} - \hat{\theta}_{11}$ with $\hat{\theta}_{11}$ being an estimate of $\theta_{11} := r^{-1}$. Differentiating both sides of (2.9) along the solutions of (2.6), (2.8), and choosing u_{1c} as

$$\begin{aligned} u_{1c} &= -k_1 x_e - \hat{\theta}_{11} f_x, \\ \dot{\hat{\theta}}_{11} &= \gamma_{11}\Omega_1^{-1} x_e f_x \end{aligned} \tag{2.10}$$

where k_1 is a positive constant, result in

$$\dot{V}_1 = -k_1\Omega_1^{-1}x_e^2 - ku_{1d}r^{-1}\Omega_1^{-2}y_e^2 + \Omega_1^{-1}x_e\tilde{u}_1 + r^{-1}\Omega_1^{-1}y_e f_y. \tag{2.11}$$

To design the virtual control u_{2c}, we take the Lyapunov function

$$V_2 = V_1 + 0.5br^{-1}z_e^2 + 0.5\gamma_{12}^{-1}\tilde{\theta}_{12}^2 + 0.5\gamma_{13}^{-1}\tilde{\theta}_{13}^2 \tag{2.12}$$

where γ_{12} and γ_{13} are positive constants, $\tilde{\theta}_{1i} = \theta_{1i} - \hat{\theta}_{1i}$, $i = 2, 3$, with $\hat{\theta}_{1i}$ being an estimate of θ_{1i}, $\theta_{12} = r^{-1}b$, $\theta_{13} = b$. Differentiating both sides of (2.12) along the solution of the third equation of (2.6), the second equation of (2.8), and choosing the virtual control u_{2c} as

$$\begin{aligned} u_{2c} &= \frac{1}{1 - k\Omega_1^{-1}x_e}(-k_2 z_e^2 - \hat{\theta}_{12} f_z - \hat{\theta}_{13} g_z u_{1c}), \\ \dot{\hat{\theta}}_{12} &= \gamma_{12} z_e f_z, \\ \dot{\hat{\theta}}_{13} &= \gamma_{13} z_e g_z u_1 \end{aligned} \tag{2.13}$$

where k_2 is a positive constant to be specified later, give

$$\begin{aligned} \dot{V}_2 = & -k_1\Omega_1^{-1}x_e^2 - ku_{1d}\Omega_1^{-2}y_e^2 - k_2 z_e^2 + r^{-1}\Omega_1^{-1}y_e f_y + \\ & (\Omega_1^{-1}x_e + \hat{\theta}_{13}z_e g_z)\tilde{u}_1 + (1 - k\Omega_1^{-1}x_e)z_e\tilde{u}_2. \end{aligned} \tag{2.14}$$

Remark 2.5. From (2.5), (2.10) and (2.13), one can consider u_{1c} and u_{2c} as an interesting combination of time-varying stabilization and tracking controllers proposed in literature, if the robot velocities are used as the actual inputs. In fact, setting $\lambda_2 = 0$ results in u_{1c} and u_{2c} similar to a tracking control law proposed in the literature. On the other hand, setting $\lambda_1 = 0$ results in u_{1c} and u_{2c} similar to a stabilization control law in the literature. However, our approach is different from those existing ones in the sense that the robot orientation error ϕ_e is used as a virtual control to stabilize the y_e-dynamics.

Step 2. Defining $\tilde{\omega}_1 = \omega_1 - \omega_{1c}$, $\tilde{\omega}_2 = \omega_2 - \omega_{2c}$ with $\omega_{1c} = u_{1c} + u_{2c}$, $\omega_{2c} = u_{1c} - u_{2c}$, and using (2.8) result in $\tilde{u}_1 = 0.5(\tilde{\omega}_1 + \tilde{\omega}_2)$, $\tilde{u}_2 = 0.5(\tilde{\omega}_1 - \tilde{\omega}_2)$. Therefore, we have $\tilde{\omega} = \omega - \omega_c$ with $\tilde{\omega} = [\tilde{\omega}_1\ \tilde{\omega}_2]^T$, $\omega_c = [\omega_{1c}\ \omega_{2c}]^T$. Differentiating both sides of $\tilde{\omega} = \omega - \omega_c$, multiplying by M and using the last equation of (2.6) result in

$$M\dot{\tilde{\omega}} = -D\tilde{\omega} + \Phi\Theta_2 + \tau \tag{2.15}$$

where the regression matrix Φ and the unknown parameter vector Θ_2 are defined as

$$\Phi = \begin{bmatrix} -\omega_1 u_2 & -\omega_{1c} & 0 & -\Omega_{11} & -\Omega_{21} & -\Omega_{12} & -\Omega_{22} & -\Omega_{13} & -\Omega_{23} \\ \omega_2 u_2 & 0 & -\omega_{2c} & -\Omega_{21} & -\Omega_{11} & -\Omega_{22} & -\Omega_{12} & -\Omega_{23} & -\Omega_{13} \end{bmatrix}, \tag{2.16}$$

$$\Theta_2 = \begin{bmatrix} \frac{r^3}{2b^2} m_c d & d_{11} & d_{22} & m_{11} & m_{12} & m_{11} r & m_{12} r & \frac{m_{11} r}{b} & \frac{m_{12} r}{b} \end{bmatrix}^T$$

with

$$\begin{aligned} \Omega_{i1} = {} & \frac{\partial \omega_{ic}}{\partial x_e} f_x + \frac{\partial \omega_{ic}}{\partial y_e}\left(-k u_{1d}\Omega_1^{-1} + f_y\right) + \frac{\partial v_{ic}}{\partial z_e} f_z + \\ & \frac{\partial \omega_{ic}}{\partial k}\dot{k} + \frac{\partial \omega_{ic}}{\partial \dot{k}}\ddot{k} + \frac{\partial v_{ic}}{\partial u_{1d}}\dot{u}_{1d} + \frac{\partial \omega_{ic}}{\partial u_{2d}}\dot{u}_{2d} + \sum_{j=1}^{3}\frac{\partial v_{ic}}{\partial \hat{\theta}_{1j}}\dot{\hat{\theta}}_{1j}, \\ \Omega_{i2} = {} & \frac{\partial \omega_{ic}}{\partial x_e} u_1 + \frac{\partial \omega_{ic}}{\partial z_e} g_z u_1, \\ \Omega_{i3} = {} & \frac{\partial \omega_{ic}}{\partial x_e} y_e u_2 - \frac{\partial \omega_{ic}}{\partial y_e} x_e u_2 + \frac{\partial \omega_{ic}}{\partial z_e}\left(1 - k\Omega_1^{-1} x_e\right) u_2 \end{aligned} \tag{2.17}$$

for $i = 1, 2$. To design the actual control input vector τ, we take the Lyapunov function

$$V_3 = V_2 + 0.5(\tilde{\omega}^T M \tilde{\omega} + \tilde{\Theta}_2^T \Gamma_2^{-1} \tilde{\Theta}_2) \tag{2.18}$$

where the adaptation gain matrix $\Gamma_2 = \Gamma_2^T$ is positive definite, $\tilde{\Theta}_2 = \Theta_2 - \hat{\Theta}_2$ with $\hat{\Theta}_2$ being an estimate vector of Θ_2. Differentiating both sides of (2.18) along the solution of (2.15) and (2.14), and choosing the actual control τ as

$$\begin{aligned} \tau &= -K_2\tilde{v} - \Phi\hat{\Theta}_2 - 0.5\begin{bmatrix} \Omega_1^{-1} x_e + \hat{\theta}_{13} z_e g_z + (1 - k\Omega_1^{-1} x_e) z_e \\ \Omega_1^{-1} x_e + \hat{\theta}_{13} z_e g_z - (1 - k\Omega_1^{-1} x_e) z_e \end{bmatrix}, \\ \dot{\hat{\Theta}}_2 &= \Gamma_2 \Phi^T \tilde{v} \end{aligned} \tag{2.19}$$

where $K_2 = K_2^T > 0$, result in

$$\begin{aligned} \dot{V}_3 = {} & -k_1\Omega_1^{-1} x_e^2 - k u_{1d}\Omega_1^{-2} y_e^2 - \\ & k_2 z_e^2 - \tilde{v}^T(D + K_2)\tilde{v} + \text{[illegible]} \end{aligned} \tag{2.20}$$

It is noted from (2.7) that $\left|r^{-1}\Omega_1^{-1}y_e f_y\right| \leq \varepsilon\Omega_1^{-2}u_{1d}^2 y_e^2 + r^{-2}\varepsilon^{-1}z_e^2$ with ε being a positive constant. Substituting $k(t)$ in (2.5) into (2.20) yields

$$\begin{aligned}\dot{V}_3 \leq\ & -k_1\Omega_1^{-1}x_e^2 - u_{1d}^2\Omega_1^{-2}(\lambda_1-\varepsilon)y_e^2 - \lambda_2\cos(\lambda_3 t)u_{1d}\Omega_1^{-2}y_e^2 - \\ & (k_2 - r^{-2}\varepsilon^{-1})z_e^2 - \tilde{\omega}^T(D+K_2)\tilde{\omega}. \end{aligned} \tag{2.21}$$

We now present the main result of this section.

Theorem 2.6. *Under Assumptions 2.1 and 2.2, there exist appropriate design constants k_2, λ_i, $i = 1,2,3$ such that the adaptive control law (2.19) forces the mobile robot (2.1) to globally asymptotically track the virtual vehicle (2.2).*

Proof. As required in (2.5), we first choose the design constants λ_1 and λ_2 such that $|k(t)| < 1$, i.e.

$$\lambda_1 u_{1d}^{\max} + \lambda_2 < 1 \tag{2.22}$$

where $u_{1d}^{\max}$ denotes the maximum value of $|u_{1d}(t)|$. From (2.21), λ_1 and k_2 are chosen such that

$$\lambda_1 > \varepsilon,\ k_2 > r_{\min}^{-2}\varepsilon^{-1} \tag{2.23}$$

where $r_{\min}$ denotes the minimum value of r. We now consider each case of Assumption 2.1.

Cases C1 and C2. From (2.21), we have

$$\begin{aligned}\dot{V}_3 \leq\ & -k_1\Omega_1^{-2}x_e^2 - u_{1d}^2\Omega_1^{-2}(\lambda_1-\varepsilon)y_e^2 + |\lambda_2 u_{1d}| - \\ & (k_2 - r^{-2}\varepsilon^{-1})z_e^2 - \tilde{\omega}^T(D+K_2)\tilde{\omega} \end{aligned} \tag{2.24}$$

By integrating both sides of (2.24), it is seen that $V_3(t) \leq \pi_3(\cdot)$ with $\pi_3(\cdot)$ being a class-K function of $\|X_e(t_0)\|$ with $X_e := [x_e, y_e, z_e, \tilde{v}, \tilde{\theta}_{1i}, \tilde{\Theta}_2]^T$. An application of Barbalat's lemma (Lemma A.25) to (2.24) shows that $\lim_{t\to\infty}(x_e(t), z_e(t), \tilde{\omega}(t)) = 0$. Also from $\tilde{u}_1 = 0.5(\tilde{\omega}_1 + \tilde{\omega}_2)$, $\tilde{u}_2 = 0.5(\tilde{\omega}_1 - \tilde{\omega}_2)$, we have $\lim_{t\to\infty}(\tilde{u}_1(t), \tilde{u}_2(t)) = 0$. Since $V_3(t) \leq \pi_3(\cdot)$, we have that $y_e(t)$,$\tilde{\theta}_{1i}$, $i = 1,2,3$ and $\tilde{\Theta}_2$ are bounded. Hence $\hat{\theta}_{1i}$, $i = 1,2,3$ and $\hat{\Theta}_2$ are bounded as well. To prove that $\lim_{t\to\infty} y_e(t) = 0$, we substitute $u_1 = u_{1c} + \tilde{u}_1$ and $u_2 = u_{2c} + \tilde{u}_2$ with u_{1c} and u_{2c} being given in (2.10) and (2.13) to the first equation of (2.6) to have

$$\dot{x}_e = -krx_e + \Omega(\cdot) \tag{2.25}$$

$$\begin{aligned}\Omega(\cdot) = & r\tilde{\theta}_{11}f_x + r\tilde{u}_1 + \frac{r}{b}y_e\tilde{u}_2 + \\ & \frac{ry_e}{b(1-k\Omega_1^{-1}x_e)}(-k_2 z_e^2 - \hat{\theta}_{12}f_z - \hat{\theta}_{13}g_z(-k_1x_e - \hat{\theta}_{11}f_x)). \end{aligned} \tag{2.26}$$

Applying Lemma A.27 to (2.25) yields $\lim_{t\to\infty}\Omega(\cdot) = 0$.

From (2.7), $\lim_{t\to\infty} u_{1d}(t) = 0$ and $\lim_{t\to\infty}(x_e(t), z_e(t), \tilde{u}_1(t), \tilde{u}_2(t)) = 0$, it is readily shown that $\lim_{t\to\infty} \Omega(\cdot) = 0$ is equivalent to

$$\lim_{t\to\infty} \left(y_e(t)\Big[-u_{2d} + (\lambda_1\dot{u}_{1d} - \frac{\lambda_2\lambda_3\sin(\lambda_3 t))y_e(t)}{\sqrt{1+(1-(\lambda_2\cos(\lambda_3 t))^2)y_e^2(t)}}\Big]\right) = 0. \quad (2.27)$$

To show that $\lim_{t\to\infty} y_e(t) = 0$ based on (2.27), we investigate **C1** and **C2** separately.

For the case **C1**, since $\lim_{t\to\infty}(u_{2d}(t), \dot{u}_{1d}(t)) = 0$, we can equivalently write (2.27) as

$$\lim_{t\to\infty}(\lambda_2\lambda_3 y_e^2\sin(\lambda_3 t)) = 0. \quad (2.28)$$

But from (2.24), we have $\frac{d}{dt}(V_3 - \int_0^t |\lambda_2 u_{1d}(\tau)|\,d\tau) \leq 0$, which implies that $V_3 - \int_0^t |\lambda_2 u_{1d}(\tau)|\,d\tau$ is non-increasing. Since V_3 is bounded from below by zero, V_3 tends to a finite nonnegative constant depending on $\|X_e(t_0)\|$. This implies that the limit of $|y_e(t)|$ exists and is finite, say l_{y_e}. If l_{y_e} were not zero, there would have existed a sequence of increasing time instants $\{t_i\}_{i=1}^{\infty}$ with $t_i \to \infty$, such that both of the limits of $\sin(\lambda_3 t_i)$ and $|y_e(t_i)|$ are not zero, which is impossible because of (2.28) for any $\lambda_2 > 0$ and $\lambda_3 > 0$. Hence l_{y_e} must be zero, i.e. $\lim_{t\to\infty} y_e(t) = 0$ in this case.

For the case **C2**, since $\lim_{t\to\infty} \dot{u}_{1d}(t) = 0$, we can equivalently write (2.27) as

$$\lim_{t\to\infty}\left(y_e(t)\left[-u_{2d} + \frac{\lambda_2\lambda_3\sin(\lambda_3 t)y_e(t)}{\sqrt{1+(1-(\lambda_2\cos(\lambda_3 t))^2)y_e^2(t)}}\right]\right) = 0. \quad (2.29)$$

Since $\frac{|y_e(t)|}{\sqrt{1+(1-(\lambda_2\cos(\lambda_3 t))^2)y_e^2(t)}} \leq \frac{1}{\sqrt{1-\lambda_2^2}}$ and $|u_{2d}(t)| \geq \mu_{22} > 0$ in this case, if we choose any $\lambda_2 > 0$ and $\lambda_3 > 0$ such that

$$\frac{\lambda_2\lambda_3}{\sqrt{1-\lambda_2^2}} < \mu_{22} \quad (2.30)$$

then (2.29) implies that $\lim_{t\to\infty} y_e(t) = 0$. It is noted again that in this case if we do not assume that u_{2d} is not of sinusoid signal, the square bracket in (2.29) might be zero at certain time instants t_i with $y_e(t_i) \neq 0$. Consequently, one cannot prove $\lim_{t\to\infty} y_e(t) = 0$ based on (2.29).

Case C3. In this case, we rewrite (2.21) as

$$\begin{aligned}\dot{V}_3 \leq\ & -k_1\Omega_1^{-1}x_e^2 - (u_{1d}^2(\lambda_1 - \varepsilon) - \lambda_2|u_{1d}|)\Omega_1^{-2}y_e^2 - \\ & (k_2 - r^{-2}\varepsilon^{-1})z_e^2 - \tilde{\omega}^T(D + K_2)\tilde{\omega}\end{aligned} \quad (2.31)$$

Therefore, if we choose the design constants λ_1 and λ_2 such that

$$u_{1d}^2(\lambda_1 - \varepsilon) - \lambda_2 |u_{1d}| \geq \mu_4 \tag{2.32}$$

where μ_4 is a strictly positive constant, (2.31) is equivalent to

$$\dot{V}_3 \leq -k_1 \Omega_1^{-1} x_e^2 - \mu_4 \Omega_1^{-2} y_e^2 - (k_2 - r^{-2}\varepsilon^{-1}) z_e^2 - \tilde{\omega}^T (D + K_2)\tilde{\omega} \tag{2.33}$$

Integrating both sides of (2.33) and using Barbalat's lemma show that $\lim_{t\to\infty}(x_e(t), y_e(t), z_e(t), \tilde{\omega}(t)) = 0$. From (2.3), one can write (2.32) as

$$\lambda_2 \leq ((\lambda_1 - \varepsilon)\mu_3^2 - \mu_4)/u_{1d}^{\max}. \tag{2.34}$$

In summary, the design constants λ_1, λ_2, λ_3 and k_2 are chosen such that (2.22), (2.23), (2.30) and (2.34) hold.

2.1.3 Simulations

In this section, we perform some numerical simulations to illustrate the effectiveness of the proposed controller in the previous section. We only do simulations for cases **C1** and **C3**. The physical parameters are given in Section 1.3.1, Chapter 1. The reference velocities are chosen as: for the case C1: $u_{1d} = u_{2d} = 0$; for the case C3: $u_{1d} = 2$, $u_{2d} = 0$ for the first 20 seconds and $u_{1d} = 2, u_{2d} = 0.1$ for the rest. The initial conditions are picked as: $(\eta^T, v^T) = ((1, 1, 0.2), (0, 0))$, $(x_d, y_d, \phi_d) = (0, 0, 0)$. We take all of initial values of the parameter estimates to be 75% of their true values. Based on Theorem 2.6, control and adaptation gains are chosen as $k_1 = 2, k_2 = 5$, $K_2 = \mathrm{diag}(2, 2)$, $\lambda_1 = 0.4$, $\lambda_2 = 0.05, \lambda_3 = 4$, $\gamma_{1i} = 2$, $i = 1, 2, 3$, $\Gamma_2 = \mathrm{diag}(2)$. It is verified that the above choice satisfies requirements in Theorem 2.6. Results are plotted in Figures 2.1 and 2.2. Slow convergence of the errors in the case C1 is a well-know effect when using smooth time-varying controllers. Note that we only plot the tracking errors and control inputs in the case **C3** for the first 30 seconds.

2.2 Simultaneous tracking and stabilization: Output feedback

2.2.1 Problem statement

We consider a mobile robot with two actuated wheels in Chapter 1. For convenience, we write the equations of motion here:

$$\begin{aligned} \dot{\eta} &= J(\eta)\omega \\ M\dot{\omega} + C(\dot{\eta})\omega + D\omega &= \tau \end{aligned} \tag{2.35}$$

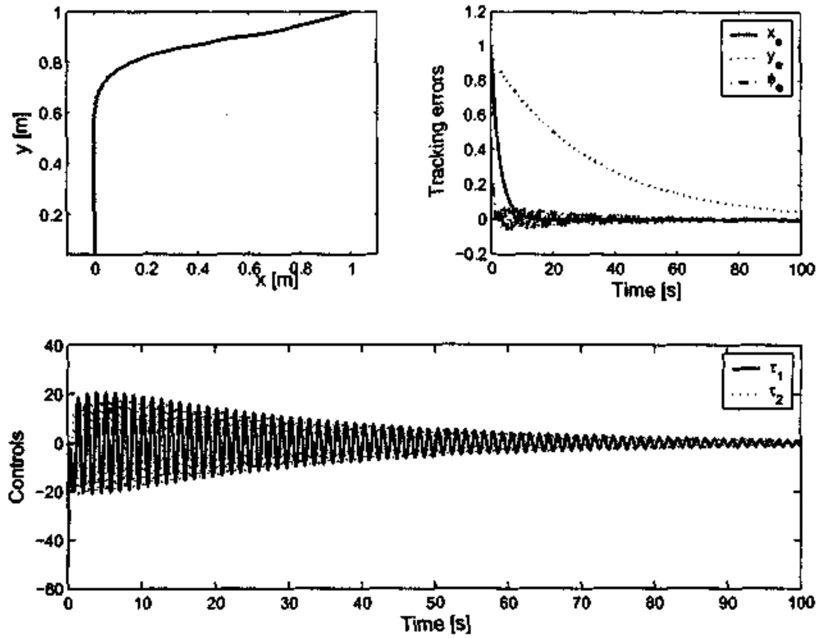


Fig. 2.1. Simulation results subject to C1. Top-left: Robot position in (x,y) plane; Top-right: Tracking errors; Bottom: Controls.

where all the state variables and parameters are defined in Section 1.3.1, Chapter 1. We assume that the reference trajectory is generated by the following virtual robot:

$$\begin{aligned} \dot{x}_d &= \cos(\phi_d)u_{1d} \\ \dot{y}_d &= \sin(\phi_d)u_{1d} \\ \dot{\phi}_d &= u_{2d} \end{aligned} \tag{2.36}$$

where (x_d, y_d, ϕ_d) are the position and orientation of the virtual robot; u_{1d} and u_{2d} are the linear and angular velocities of the virtual robot, respectively.

Control objective: Under Assumption 2.7, design the control input vector τ to force the position and orientation, (x, y, ϕ) of the real robot (2.35) to globally asymptotically track (x_d, y_d, ϕ_d) generated by (2.36) with only (x, y, ϕ) available for feedback.

Assumption 2.7 *The reference signals u_{1d}, $\dot{u}_{1d}$, $\ddot{u}_{1d}, u_{2d}$ and $\dot{u}_{2d}$ are bounded. In addition, one of the following conditions holds:*

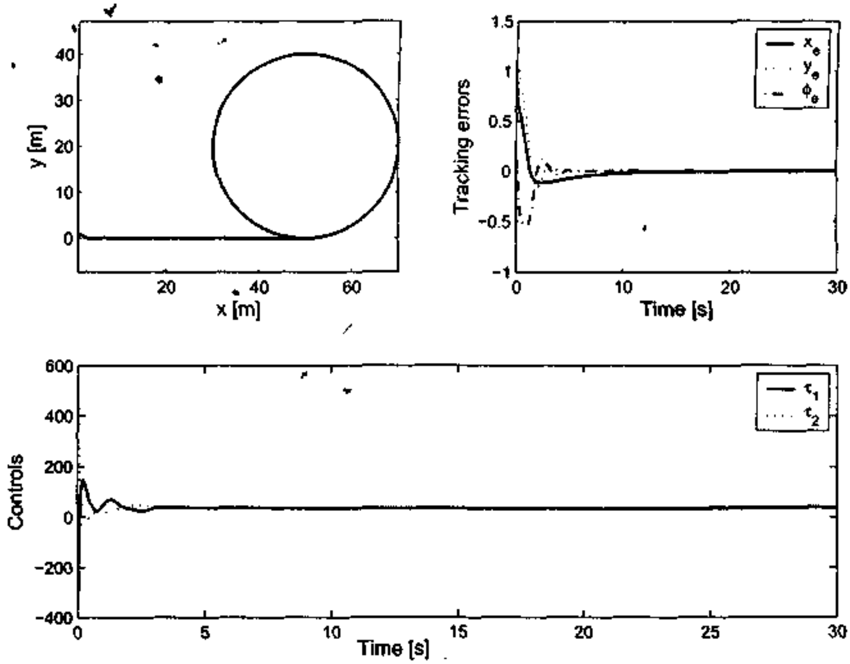


Fig. 2.2. Simulation results subject to C3. Top-left: Robot position in (x,y) plane; Top-right: Tracking errors; Bottom: Controls.

$$C1.\ \int_0^\infty \left(|u_{1d}(t)| + |u_{2d}(t)|\right) dt \le \mu_{11}$$

$$C2.\ \int_0^\infty |u_{1d}(t)|\, dt \le \mu_{21} \ \textit{and}\ \ |u_{2d}(t)| \ge \mu_{22}$$

$$C3.\ \int_{t_0}^{t} u_{1d}^2(\tau) d\tau \ge \mu_{31}(t - t_0) - \mu_{32},\ \forall t \ge t_0 \ge 0 \tag{2.37}$$

where μ_{11}, μ_{21} *and* μ_{32} *are nonnegative constants;* μ_{22} *and* μ_{31} *are strictly positive constant.*

2) The robot wheel velocities $\omega = [\omega_1 \quad \omega_2]^T$ *are not available for feedback.*

Remark 2.8. All remarks followed by Assumption 2.1 hold for the item 1) of Assumption 2.7. Moreover, the item 2) of Assumption 2.7 implies that we need to design an output feedback controller.

2.2.2 Observer design

We first remove the quadratic velocity terms in the mobile robot dynamics by introducing the following coordinate change:

$$X = Q(\eta)\omega \tag{2.38}$$

where $Q(\eta)$ is a globally invertible matrix with bounded elements to be determined. Using (2.38), we write the second equation of (2.35) as follows:

$$\dot{X} = [\dot{Q}(\eta)\omega - Q(\eta)M^{-1}C(\dot{\eta})\omega] + Q(\eta)M^{-1}(-D\omega + \tau) \tag{2.39}$$

In [11], the author requires $Q(\eta)$ with the above properties such that $\dot{Q}(\eta) = Q(\eta)M^{-1}C(\dot{\eta})$, $\forall \eta \in \mathbb{R}^3$, which does not exist as a simple calculation shows.

Our method is to cancel the square bracket in the right hand side of (2.39) for all $(\eta, \omega) \in \mathbb{R}^5$. We assume that $q_{ij}(\eta)$, $i = 1, 2$, $j = 1, 2$ are the elements of $Q(\eta)$. Using the first equation of (2.35), it is readily shown that the above square bracket is zero for all $(\eta, \omega) \in \mathbb{R}^5$ if

$$\begin{aligned}
&\frac{\partial q_{i1}}{\partial x}\cos(\phi) + \frac{\partial q_{i1}}{\partial y}\sin(\phi) + \frac{\partial q_{i1}}{\partial \phi}\frac{1}{b} + \frac{n_{12}c}{b}q_{i1} + \frac{n_{11}c}{b}q_{i2} = 0,\\
&\frac{\partial q_{i2}}{\partial x}\cos(\phi) + \frac{\partial q_{i2}}{\partial y}\sin(\phi) - \frac{\partial q_{i2}}{\partial \phi}\frac{1}{b} + \frac{n_{11}c}{b}q_{i1} + \frac{n_{12}c}{b}q_{i2} = 0,\\
&\left(\frac{\partial q_{i1}}{\partial x} + \frac{\partial q_{i2}}{\partial x}\right)\cos(\phi) + \left(\frac{\partial q_{i1}}{\partial y} + \frac{\partial q_{i2}}{\partial y}\right)\sin(\phi) +\\
&\qquad\left(\frac{\partial q_{i2}}{\partial \phi} - \frac{\partial q_{i1}}{\partial \phi}\right)\frac{1}{b} - (n_{11} + n_{12})\frac{c}{b}(q_{i1} + q_{i2}) = 0
\end{aligned} \tag{2.40}$$

Using the characteristic method to solve the above partial differential equations gives a family of solutions

$$\begin{aligned}
q_{i1} &= C_{i1}\sin(c\Delta\phi) + C_{i2}\cos(c\Delta\phi),\\
q_{i2} &= n_{11}^{-1}((C_{i2}\Delta - C_{i1}n_{12})\sin(c\Delta\phi) - (C_{i1}\Delta + C_{i2}n_{12})\cos(c\Delta\phi))
\end{aligned} \tag{2.41}$$

where $i = 1, 2$, $n_{11} = m_{11}(m_{11}^2 - m_{12}^2)^{-1}, n_{12} = -m_{12}(m_{11}^2 - m_{12}^2)^{-1}, \Delta = \sqrt{n_{11}^2 - n_{12}^2}$; C_{i1} and C_{i2} are arbitrary constants. A choice of $C_{11} = C_{22} = 0$, $C_{12} = C_{21} = n_{11}$ results in

$$Q(\eta) = \begin{bmatrix} n_{11}\cos(a\Delta\phi) & \Delta\sin(a\Delta\phi) - n_{12}\cos(a\Delta\phi) \\ n_{11}\sin(a\Delta\phi) & -n_{12}\sin(a\Delta\phi) - \Delta\cos(a\Delta\phi) \end{bmatrix} \tag{2.42}$$

This matrix is globally invertible and its elements are bounded. Now we write (2.35) in the (η, X) coordinates as

$$\begin{aligned}\dot{\eta} &= J(\eta)Q^{-1}(\eta)X \\ \dot{X} &= -D_\eta(\eta)X + Q(\eta)M^{-1}\tau\end{aligned} \tag{2.43}$$

where $D_\eta(\eta) = Q(\eta)M^{-1}DQ^{-1}(\eta)$. It is seen that (2.43) is linear in the unmeasured states. Indeed a reduced-order observer can be designed but it is often noise sensitive. We here use the following passive observer:

$$\begin{aligned}\dot{\hat{\eta}} &= J(\eta)Q^{-1}(\eta)\hat{X} + K_{01}(\eta - \hat{\eta}) \\ \dot{\hat{X}} &= -D_\eta(\eta)\hat{X} + Q(\eta)M^{-1}\tau + K_{02}(\eta - \hat{\eta})\end{aligned} \tag{2.44}$$

where $\hat{\eta}$ and $\hat{X}$ are the estimates of η and X, respectively. The observer gain matrices K_{01} and K_{02} are chosen such that $Q_{01} = K_{01}^T P_{01} + P_{01}K_{01}$ and $Q_{02} = D_\eta^T(\eta)P_{02} + P_{02}D_\eta(\eta)$ are positive definite and

$$(J(\eta)Q^{-1}(\eta))^T P_{01} - P_{02}K_{02} = 0 \tag{2.45}$$

with P_{01} and P_{02} being positive definite matrices. Since $D_\eta(\eta)$ is positive definite, K_{01} and K_{02} always exist. From (2.44) and (2.43), we have

$$\begin{aligned}\dot{\tilde{\eta}} &= J(\eta)Q^{-1}(\eta)\tilde{X} - K_{01}\tilde{\eta}, \\ \dot{\tilde{X}} &= -D_\eta(\eta)\tilde{X} - K_{02}\tilde{\eta}\end{aligned} \tag{2.46}$$

where $\tilde{\eta} = \eta - \hat{\eta}$ and $\tilde{X} = X - \hat{X}$. It is now seen that (2.46) is globally exponentially stable by taking the Lyapunov function $V_0 = \tilde{\eta}^T P_{01}\tilde{\eta} + \tilde{X}^T P_{02}\tilde{X}$ whose derivative along the solution of (2.46) and using (2.45) satisfies $\dot{V}_0 = -\tilde{\eta}^T Q_{01}\tilde{\eta} - \tilde{X}^T Q_{02}\tilde{X}$, which in turn implies that there exists a strictly positive constant σ_0 such that

$$\left\|(\tilde{\eta}(t), \tilde{X}(t))\right\| \le \left\|(\tilde{\eta}(t_0), \tilde{X}(t_0))\right\| e^{-\sigma_0(t-t_0)}, \ \forall t \ge t_0 \ge 0. \tag{2.47}$$

Define $\hat{\omega} = [\hat{\omega}_1 \ \hat{\omega}_2]^T$ being an estimator of the velocity vector ω as

$$\hat{\omega} = Q^{-1}(\eta)\hat{X}. \tag{2.48}$$

The velocity estimate error vector, $\tilde{\omega} = \omega - \hat{\omega}$ satisfies

$$\tilde{\omega} = Q^{-1}(\eta)\tilde{X}. \tag{2.49}$$

To prepare for the control design in the next section, we convert the wheel velocities ω_1 and ω_2 to the linear, v, and angular, w, velocities of the robot by the relationship:

$$[v\ w]^T = B^{-1}[\omega_1\ \omega_2]^T,\ \text{ with }\ B = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}. \tag{2.50}$$

By defining $\tilde{v} = v - \hat{v}$, $\tilde{w} = w - \hat{w}$ with $\hat{v}$ and $\hat{w}$ being estimates of v and w, we can see from (2.49) and (2.50) that

$$\|(\tilde{v}(t),\ \tilde{w}(t))\| \le \gamma_0 \left\|(\tilde{\eta}(t_0), \tilde{X}(t_0))\right\| e^{-\sigma_0(t-t_0)},\ \forall t \ge t_0 \ge 0 \tag{2.51}$$

where γ_0 is a positive constant. We now write (2.35) in conjunction with (2.48) and (2.50) as

$$\begin{aligned} \dot{x} &= \cos(\phi)\hat{v} + \cos(\phi)\tilde{v} \\ \dot{y} &= \sin(\phi)\hat{v} + \sin(\phi)\tilde{v} \\ \dot{\phi} &= \hat{w} + \tilde{w} \\ \dot{\hat{v}} &= \tau_{vc} + \Omega_v \\ \dot{\hat{w}} &= \tau_{wc} + \Omega_w \end{aligned} \tag{2.52}$$

where Ω_v and Ω_w are the first and second rows of Ω:

$$\Omega = B^{-1}N_cB\begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix}\tilde{w} + B^{-1}Q^{-1}(\eta)K_{02}\tilde{\eta},\ N_c = c\begin{bmatrix} n_{12} & -n_{11} \\ n_{11} & -n_{12} \end{bmatrix} \tag{2.53}$$

and we have chosen the control torque

$$\tau = MB\left(B^{-1}N_cB\begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix}\hat{w} - B^{-1}M^{-1}DB\begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} \tau_{vc} \\ \tau_{wc} \end{bmatrix}\right) \tag{2.54}$$

with τ_{vc} and τ_{wc} being the new control inputs to be designed in the next section.

2.2.3 Control design

Similar to the state feedback case, we first interpret the tracking errors as

$$\begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x - x_d \\ y - y_d \\ \phi - \phi_d \end{bmatrix}. \tag{2.55}$$

Using (2.55), (2.36) and the kinematic part of (2.52), we have the kinematic tracking errors:

$$\begin{aligned} \dot{x}_e &= \hat{v} - u_{1d}\cos(\phi_e) + y_e(\hat{w} + \tilde{w}) + \tilde{v}, \\ \dot{y}_e &= u_{1d}\sin(\phi_e) - x_e(\hat{w} + \tilde{w}), \\ \dot{\phi}_e &= \hat{w} - u_{2d} + \tilde{w}. \end{aligned} \tag{2.56}$$

Since (2.56) and the last two equations of (2.52) are of the lower triangular structure, we use the backstepping technique [12] to design τ_{vc} and τ_{wc} in two steps.

Step 1. In this step, we consider $\hat{v}$ and $\hat{w}$ as the controls. From (2.56), it is seen that $\hat{v}$ and $\hat{w}$ can be directly used to stabilize x_e and ϕ_e-dynamics. To stabilize y_e-dynamics, ϕ_e can be used when u_{1d} is PE. When u_{1d} is not PE (stabilization/regulation case), we need some PE signal in $\hat{w}$ to stabilize y_e-dynamics via x_e. With these observations in mind, we define

$$\bar{v} = \hat{v} - \alpha_v,\ \bar{w} = \hat{w} - \alpha_w,\ \bar{\phi}_e = \phi_e - \alpha_{\phi e} \tag{2.57}$$

where α_v, α_w and $\alpha_{\phi e}$ are the virtual controls of $\hat{v}$, $\hat{w}$ and ϕ_e, respectively. From the above discussion, we first choose the virtual controls α_v and $\alpha_{\phi e}$as

$$\begin{aligned}
\alpha_v &= -c_1\Omega_1^{-1}x_e + u_{1d}\cos(\phi_e),\\
\alpha_{\phi e} &= -\arcsin(k(t)\Omega_1^{-1}y_e),\\
k(t) &= \lambda_1 u_{1d} + \lambda_2\cos(\lambda_3 t)
\end{aligned} \tag{2.58}$$

where $\Omega_1 = \sqrt{1 + x_e^2 + y_e^2}$; c_1 is a positive constant; λ_i, $i = 1, 2, 3$ are positive constants such that $|k(t)| \leq k_* < 1$, $\forall t$. They will be specified later. For simplification, the virtual control α_v does not cancel a known term $y_e\hat{w}$ in the x_e-dynamics. It is of interest to note that the choice of (2.58) will result in global result and bounded virtual velocity controls.

To design α_w, differentiating $\bar{\phi}_e = \phi_e - \alpha_{\phi e}$ along the solution of (2.56) together with (2.58) yields

$$\begin{aligned}
\dot{\bar{\phi}}_e = {}& \left(1 - k\Omega_2^{-1}x_e\right)(\alpha_w + \bar{w} + \tilde{w}) - u_{2d} - k\Omega_2^{-1}\Omega_1^{-2}x_e y_e(\bar{v} + \tilde{v}) +\\
& \Omega_2^{-1}\left(\dot{k}y_e + kc_1\Omega_1^{-3}x_e^2 y_e + ku_{1d}\Omega_1^{-2}(1 + x_e^2)\sin(\phi_e)\right)
\end{aligned} \tag{2.59}$$

which suggests that we choose

$$\begin{aligned}
\alpha_w = {}& \frac{1}{1 - k\Omega_2^{-1}x_e}\Big(- \frac{c_2\bar{\phi}_e}{\sqrt{1 + \bar{\phi}_e^2}} + u_{2d} - \Omega_2^{-1}(\dot{k}y_e + kc_1\Omega_1^{-3}x_e^2 y_e +\\
& ku_{1d}\Omega_1^{-2}(1 + x_e^2)\sin(\phi_e)\Big)
\end{aligned} \tag{2.60}$$

where $\Omega_2 = \sqrt{1 + x_e^2 + (1 - k^2)y_e^2}$; c_2 is a positive constant.

Remark 2.9. From (2.58) and (2.60), the virtual controls α_v and α_w, as a simple calculation shows, are bounded by some constants depending on the upper bound of u_{1d}, $\dot{u}_{1d}$ and u_{2d}.

Substituting (2.58) and (2.60) into (2.56) and (2.59) results in

$$\begin{aligned}
\dot{x}_e &= -c_1\Omega_1^{-1}x_e + y_e(\hat{w}+\tilde{w}) + \bar{v} + \tilde{v},\\
\dot{y}_e &= -k\Omega_1^{-1}u_{1d}y_e - x_e(\hat{w}+\tilde{w}) +\\
&\quad u_{1d}\Omega_1^{-1}\left(\sin(\bar{\phi}_e)\Omega_2 - (\cos(\bar{\phi}_e)-1)ky_e\right),\\
\dot{\bar{\phi}}_e &= -\frac{c_2\bar{\phi}_e}{\sqrt{1+\bar{\phi}_e^2}} + \left(1-k\Omega_2^{-1}x_e\right)(\bar{w}+\tilde{w}) -\\
&\quad k\Omega_2^{-1}\Omega_1^{-2}x_e y_e(\bar{v}+\tilde{v})
\end{aligned} \tag{2.61}$$

Step 2. At this step, the control inputs τ_{vc} and τ_{wc} are designed. We note that α_v is a smooth function of x_e, y_e, ϕ_e and u_{1d}, and that α_w is a smooth function of x_e, y_e, ϕ_e, u_{1d}, $\dot{u}_{1d}$, u_{2d} and t. By differentiating $\bar{v} = \hat{v} - \alpha_v$ and $\bar{w} = \hat{w} - \alpha_w$ along the solution of (2.56) and the last two equations of (2.52), and noting the last equation of (2.61), we choose τ_{vc} and τ_{wc}as

$$\begin{aligned}
\tau_{vc} = &-c_3\bar{v} + \frac{\partial\alpha_v}{\partial x_e}\left(\hat{v} - u_{1d}\cos(\phi_e) + y_e\hat{w}\right) + \frac{\partial\alpha_v}{\partial\phi_e}(\hat{w}-u_{2d}) +\\
&\frac{\partial\alpha_v}{\partial y_e}\left(u_{1d}\sin(\phi_e) - x_e\hat{w}\right) + \frac{\partial\alpha_v}{\partial u_{1d}}\dot{u}_{1d} - \delta_v\left(\hat{v}^2+\hat{w}^2\right)\bar{v} +\\
&k\Omega_2^{-1}\Omega_1^{-2}x_e y_e\bar{\phi}_e,\\
\tau_{wc} = &-c_4\bar{w} + \frac{\partial\alpha_w}{\partial x_e}\left(\hat{v} - u_{1d}\cos(\phi_e) + y_e\hat{w}\right) + \frac{\partial\alpha_w}{\partial\phi_e}(\hat{w}-u_{2d}) +\\
&\frac{\partial\alpha_w}{\partial y_e}\left(u_{1d}\sin(\phi_e) - x_e\hat{w}\right) + \frac{\partial\alpha_w}{\partial u_{1d}}\dot{u}_{1d} + \frac{\partial\alpha_w}{\partial\dot{u}_{1d}}\ddot{u}_{1d} + \frac{\partial\alpha_w}{\partial u_{2d}}\dot{u}_{2d} -\\
&\left(1-k\Omega_2^{-1}x_e\right)\bar{\phi}_e - \delta_w\left(\hat{v}^2+\hat{w}^2\right)\bar{w}
\end{aligned} \tag{2.62}$$

where c_3, c_4, δ_v and δ_w are positive constants. The terms multiplied by δ_v and δ_w are the nonlinear damping terms to overcome the effect of observer errors, see (2.53). The choice of (2.62) results in

$$\begin{aligned}
\dot{\bar{v}} = &-c_3\bar{v} - \frac{\partial\alpha_v}{\partial x_e}(y_e\tilde{w}+\tilde{v}) - \frac{\partial\alpha_v}{\partial\phi_e}\tilde{w} + \frac{\partial\alpha_v}{\partial y_e}x_e\tilde{w} +\\
&\Omega_v - \delta_v\left(\hat{v}^2+\hat{w}^2\right)\bar{v} + k\Omega_2^{-1}\Omega_1^{-2}x_e y_e\bar{\phi}\\
\dot{\bar{w}} = &-c_4\bar{w} - \frac{\partial\alpha_w}{\partial x_e}(y_e\tilde{w}+\tilde{v}) - \frac{\partial\alpha_w}{\partial\phi_e}\tilde{w} + \frac{\partial\alpha_w}{\partial y_e}x_e\tilde{w} +\\
&\Omega_w - \left(1-k\Omega_2^{-1}x_e\right)\bar{\phi}_e - \delta_w\left(\hat{v}^2+\hat{w}^2\right)\bar{w}.
\end{aligned} \tag{2.63}$$

To analyze the closed loop consisting of (2.61) and (2.63), we first consider the $(\bar{\phi}_e, \bar{v}, \bar{w})$-subsystem then move to (x_e, y_e)-subsystem.

$(\bar{\phi}_e, \bar{v}, \bar{w})$ -subsystem

For this subsystem, consider the Lyapunov function

$$V_1 = 0.5(\bar{\phi}_e^2 + \bar{v}^2 + \bar{w}^2) \tag{2.64}$$

whose derivative along the solution of the last equation of (2.61) and (2.63) satisfies

$$\begin{aligned} \dot{V}_1 &\le -c_2\bar{\phi}_e^2/\sqrt{1+\bar{\phi}_e^2} - c_3\bar{v}^2 - c_4\bar{w}^2 + (\chi_{11} + \chi_{12}V_1)\, e^{-\sigma_0(t-t_0)} \\ &\le (\chi_{11}V_1 + \chi_{12})\, e^{-\sigma_0(t-t_0)} \end{aligned} \tag{2.65}$$

where χ_{11} and χ_{12} are class-K functions of $\left\|(\tilde{\eta}(t_0), \bar{X}(t_0))\right\|$. The second line of (2.65) implies that $V_1(t) \le \chi_{13}$ with χ_{13} being a class-K function of $\left\|(\tilde{\eta}(t_0), \tilde{X}(t_0), \bar{X}(t_0))\right\|$ with $\bar{X}(t) = \left[\bar{\phi}_e(t)\ \bar{v}(t)\ \bar{w}(t)\right]^T$. Substituting this bound into the first line of (2.65) yields

$$\dot{V}_1 \le -2\min\left(c_2/\sqrt{1+2\chi_{13}}, c_3, c_4\right) V_1 + (\chi_{11} + \chi_{12}\chi_{13})\, e^{-\sigma_0(t-t_0)} \tag{2.66}$$

which implies that there exist $\sigma_1 > 0$ and a class-K function χ_1 depending on $\left\|(\tilde{\eta}(t_0), \tilde{X}(t_0), \bar{X}(t_0))\right\|$ such that $\|\bar{X}(t)\| \le \chi_1 e^{-\sigma_1(t-t_0)}$, i.e. the $(\bar{\phi}_e, \bar{v}, \bar{w})$-subsystem is globally asymptotically stable.

(x_e, y_e)-subsystem

We first prove that the trajectories (x_e, y_e) are bounded by taking the Lyapunov function

$$V_2 = \sqrt{1 + x_e^2 + y_e^2} - 1 \tag{2.67}$$

whose derivative along the solution of the first two equations of (2.61) satisfies

$$\begin{aligned} \dot{V}_2 &\le -c_1\Omega_1^{-2}x_e^2 - ku_{1d}\Omega_1^{-2}y_e^2 + \chi_{21}e^{-\sigma_{21}(t-t_0)} \\ &\le \lambda_2 u_{1d}\cos(\lambda_3 t)\Omega_1^{-2}y_e^2 + \chi_{21}e^{-\sigma_{21}(t-t_0)} \end{aligned} \tag{2.68}$$

where $\sigma_{21} = \min(\sigma_0, \sigma_1)$ and χ_{21} is a class-K function of $\left\|(\tilde{\eta}(t_0), \tilde{X}(t_0), \bar{X}(t_0))\right\|$. Integrating both sides of the second line of (2.68) yields

$$V_2(t) \le V_2(t_0) + 2\lambda_2 u_{1d}^{\max} + \chi_{21}/\sigma_{21} \le \chi_{22} \tag{2.69}$$

where $u_{1d}^{\max}$ is the upper bound of $|u_{1d}(t)|$. Therefore, the trajectories (x_e, y_e) are bounded on $[0, \infty)$. To prove convergence of (x_e, y_e) to zero, we consider each case of Assumption 2.7.

Cases C1 and C2. From the first line of (2.68) and noting (2.58), we have

$$\dot{V}_2 \le -c_1\Omega_1^{-2}x_e^2 + |\lambda_2 u_{1d}| + \chi_{21}(\cdot)e^{-\sigma_{21}(t-t_0)}. \tag{2.70}$$

Integrating both sides of (2.70) and using Barbalat's lemma, we have $\lim_{t\to\infty} x_e(t) = 0$. To prove that $\lim_{t\to\infty} y_e(t) = 0$, applying Lemma A.27 to the first equation of (2.61) yields:

$$\lim_{t\to\infty} (y_e(\alpha_w + \bar{w} + \tilde{w}) + \tilde{v} + \bar{v}) = 0 \tag{2.71}$$

which is equivalent to:

$$\lim_{t\to\infty} \Xi(t) = 0 \tag{2.72}$$

where

$$\Xi(t) = y_e(t)\left(\dot{k}(t)y_e(t)/\sqrt{1+(1-k^2(t))y_e^2(t)} - u_{2d}(t)\right) \tag{2.73}$$

On the other hand from (2.70), we have

$$\frac{d}{dt}(V_2 - \int_0^t |\lambda_2 u_{1d}(\tau)|\,d\tau + \sigma_{21}^{-1}\chi_{21}(\cdot)e^{-\sigma_{21}(t-t_0)}) \leq 0 \tag{2.74}$$

which means that $V_2 - \int_0^t |\lambda_2 u_{1d}(\tau)|\,d\tau + \sigma_{21}^{-1}\chi_{21}(\cdot)e^{-\sigma_{21}(t-t_0)}$ is non-increasing. Since V_2 is bounded from below by zero, V_2 tends to a finite nonnegative constant depending on $\|\bar{X}_e(t_0)\|$ with $\bar{X}_e = (x_e, y_e, \bar{X}, \tilde{\eta}(t_0), \tilde{X}(t_0))$. This implies that the limit of $|y_e(t)|$ exists and is finite, say l_{y_e}. If l_{y_e} was not zero, there would exist a sequence of increasing time instants $\{t_i\}_{i=1}^{\infty}$ with $t_i \to \infty$, such that both of the limits of $\dot{k}(t_i)$ and $\Xi(t_i)$ are not zero. With this in mind, if we choose $\lambda_i \neq 0$ and such that

$$\lambda_2\lambda_3/\sqrt{1-k_*^2} < \mu_{22} \tag{2.75}$$

then under Conditions (2.37), $\dot{k}(t_i)$ and $\Xi(t_i)$ cannot be nonzero simultaneously for any t_i. Hence l_{y_e} must be zero, which allows us to conclude from (2.72) that $\lim_{t\to\infty} y_e(t) = 0$,i.e. the (x_e, y_e)-subsystem is asymptotically stable.

Case C3. In this case, from (2.68), we have

$$\dot{V}_2 \leq -\Omega_1^{-2}(c_1 x_e^2 + (\lambda_1 u_{1d}^2 - \lambda_2 |u_{1d}|)y_e^2) + \chi_{21}e^{-\sigma_{21}(t-t_0)} \tag{2.76}$$

which means that there exist $\sigma_2 > 0$ and a class-K function χ_2 depending on $\|\bar{X}(t_0)\|$ such that

$$\|(x_e(t), y_e(t))\| \leq \chi_2\sigma^{-\sigma_2(t-t_0)} \tag{2.77}$$

as long as

$$\lambda_1\mu_{31} - \lambda_2 u_{1d}^{\max} \geq \mu_{31}^* \tag{2.78}$$

where μ_{31}^* is a positive constant. In addition, it can be shown that in this case the closed loop of (2.61) and (2.63) is also locally exponential stable. Under Assumption 1, there always exist λ_i such that (2.75) and (2.78). We have thus proven the following result.

Theorem 2.10. *Under Assumption 2.7, the output-feedback control laws consisting of (2.54) and (2.62) force the mobile robot (2.35) to globally asymptotically track the virtual vehicle (2.36) if the constants λ_i, $i = 1, 2, 3$ are chosen such that $\lambda_i \neq 0$, (2.75) and (2.78) hold.*

2.2.4 Simulations

The physical parameters are taken from Section 1.3.1, Chapter 1. We perform two simulations. For the first simulation, the reference velocities are chosen as: $u_{1d} = 0.5(\tanh(t_s - t) + 1), u_{2d} = 0$, where t_s is a positive constant. A switching combination of a tracking controller and a stabilization one in the literature cannot be used to fulfill this task if t_s is unknown in advance. A calculation shows that for $t \leq t_s$(tracking a curve), condition C3 holds with $\mu_{31} = 0.25$ and for $t > t_s$ (parking) C1 holds. Hence, our proposed controller can be applied. We also assume that due to some sudden impact at the time $t_m > t_s$, the robot position is perturbed to $y = y_m \neq 0$ to illustrate the regulation ability of our proposed controller. For the second simulation, the reference velocities are $u_{1d} = 0, u_{2d} = 0.2$, i.e. C2 holds with $\mu_{22} = 0.2$. The initial conditions are: $(\eta^T, \omega^T) = ((-2, 2, -0.5), (0, 0)), (\hat{\eta}^T, \hat{X}^T) = ((0, 0, 0), (0, 0)), (x_d, y_d, \phi_d) = (0, 0, 0)$, and we take $t_s = 20, t_m = 30, y_m = 1.5$. The control and observer gains are chosen as $c_i = 2,\ 1 \leq i \leq 4, \delta_v = \delta_\omega = 0.1,\ P_{01} = P_{02} = \text{diag}(1, 1), \lambda_1 = \lambda_3 = 0.5,\ \lambda_2 = 0.1, K_{01} = \text{diag}(1, 1), K_{02} = (J(\eta)Q^{-1}(\eta))^T$. The above choice satisfies requirements in Theorem 2.10. Results are plotted in Figures 2.3 and 2.4 (robot position in (x,y) plane). The tracking errors in the form of $\sqrt{x_e^2 + y_e^2 + \phi_e^2}$ are plotted in Figure 2.5. This figure indicates that convergence of the tracking errors for the case of regulation to zero is much slower than for the other cases, which is a quite well-known effect when using the smooth time-varying controllers. Convergence of tracking errors in the case of C2 is slower than that in the case of C3, since C3 yields local exponential stability but only asymptotic for C2 (see proof of Theorem 2.10).

2.3 Path following

2.3.1 Problem statement

We consider a mobile robot with two actuated wheels in Chapter 1. For convenience, we write the equations of motion here:

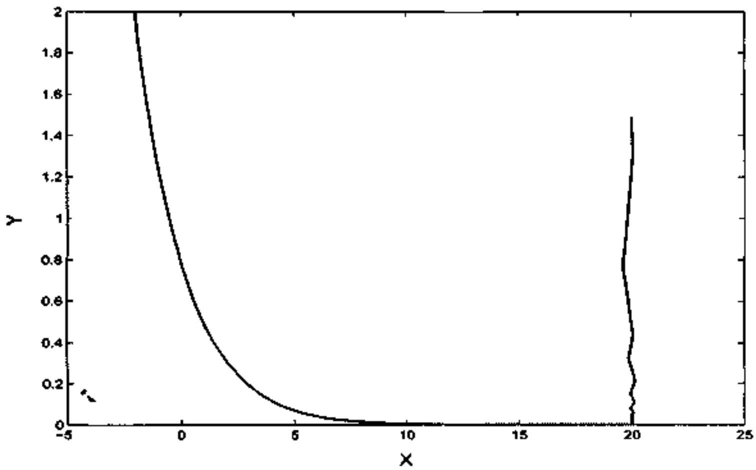


Fig. 2.3. First simulation: Robot position in (x, y) plane.

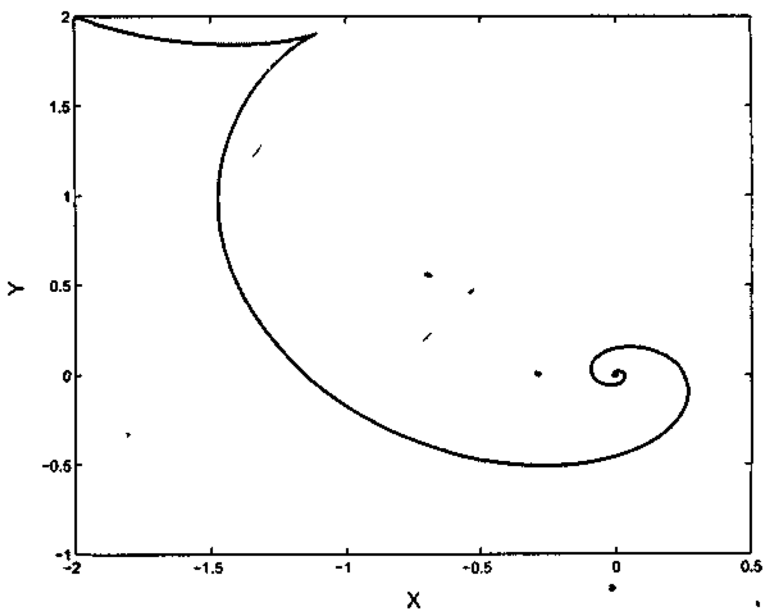


Fig. 2.4. Second simulation: Robot position in (x, y) plane.

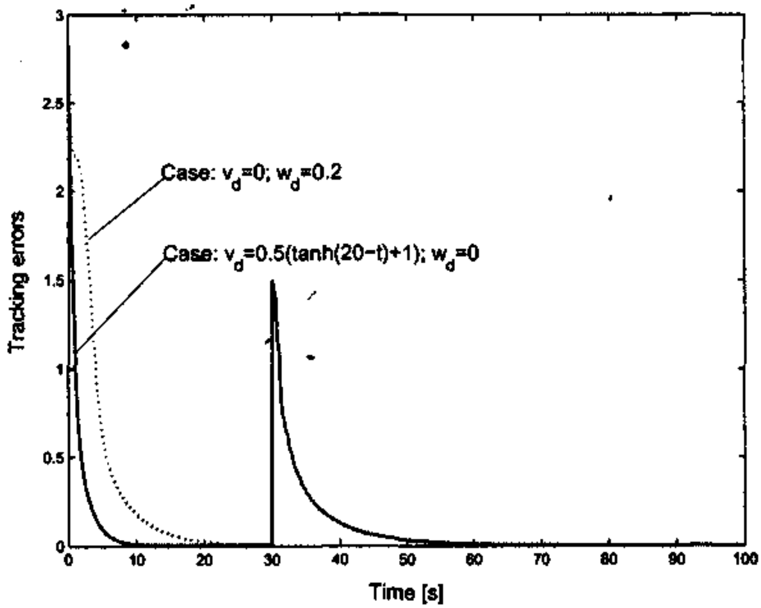


Fig. 2.5. Tracking errors with respect to the first and second simulations.

$$\begin{aligned} \dot{\eta} &= J(\eta)\omega \\ M\dot{\omega} + C(\dot{\eta})\omega + D\omega &= \tau \end{aligned} \tag{2.79}$$

where all the state variables and parameters are defined in Section 1.3.1, Chapter 1. After an observer and a primary control design as in Section 2.2.2, we only need to consider the robot model (i.e. the dynamics (2.52)):

$$\begin{aligned} \dot{x} &= \cos(\phi)\hat{v} + \cos(\phi)\tilde{v} \\ \dot{y} &= \sin(\phi)\hat{v} + \sin(\phi)\tilde{v} \\ \dot{\phi} &= \hat{w} + \tilde{w} \\ \dot{\hat{v}} &= \tau_{vc} + \Omega_v \\ \dot{\hat{w}} &= \tau_{wc} + \Omega_w \end{aligned} \tag{2.80}$$

In this section, we consider a control objective of designing the control vector τ to force the mobile robot to follow a specified path Γ, see Figure 2.6. If we are able to drive the robot to follow closely a virtual robot that moves along the path with a desired speed v_0, which is tangential to the path, then the control objective is fulfilled, i.e. the robot is in a tube of nonzero diameter centered on the reference path and moves along the specified path at the speed v_0. Roughly speaking, the approach is to steer the robot such that it heads to the virtual and eliminates the distance between itself and the virtual robot. We define the following variables to mathematically formulate the control

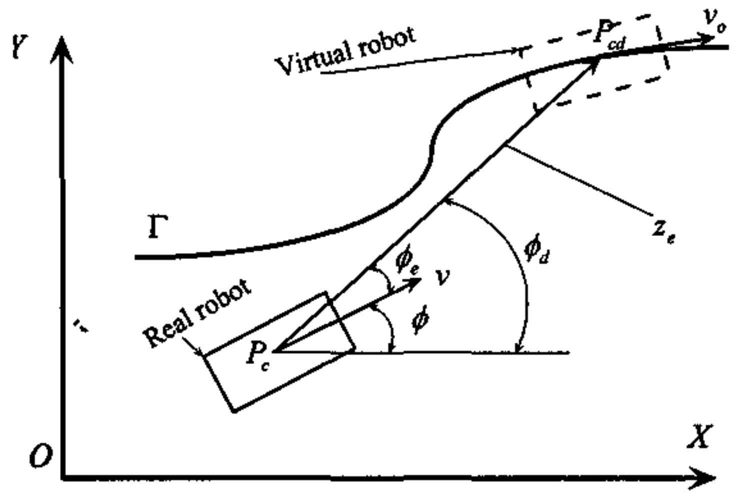


Fig. 2.6. General framework of mobile robot path following.

objective:

$$\begin{aligned} x_e &= x_d - x, \\ y_e &= y_d - x, \\ \phi_e &= \phi - \phi_d, \\ z_e &= \sqrt{x_e^2 + y_e^2} \end{aligned} \tag{2.81}$$

where

$$\phi_d = \arcsin(y_e/z_e). \tag{2.82}$$

Control objective: Under Assumption 2.11, design the controls τ_1 and τ_2 to force the mobile robot (2.80) to follow the path Γ given by

$$\begin{aligned} x_d &= x_d(s), \\ y_d &= y_d(s) \end{aligned} \tag{2.83}$$

where s is the path parameter variable, such that

$$\begin{aligned} &\lim_{t\to\infty} z_e(t) \le \bar{z}_e, \\ &\lim_{t\to\infty} |\phi_e(t)| = 0 \end{aligned} \tag{2.84}$$

with $\bar{z}_e$ being arbitrarily a small positive constant.

Assumption 2.11 *a) The reference path is regular, i.e. there exist positive constants $R_{\min}$ and $R_{\max}$ such that*

$$0 < R_{\min} \leq \left(\frac{\partial x_d}{\partial s}\right)^2 + \left(\frac{\partial y_d}{\partial s}\right)^2 \leq R_{\max} < \infty.$$

b) The minimum radius of the osculating circle of the path is larger than or equal to the minimum possible turning radius of the robot.

Remark 2.12. a) Assumption 2.11 ensures that the path is feasible for the robot to follow.

b) If the reference path is not regular, then we can often split it into regular pieces and consider each of them separately.

c) The path parameter, s, is not the arclength of the path in general. For example, a circle with radii of R centered at the origin can be described as $x_d = R\cos(s)$, $y_d = R\sin(s)$, see [13] for more details.

If one differentiates both sides of $\phi_e = \phi - \phi_d$ to get the $\dot{\phi}_e$-dynamics, there will be discontinuity in the $\dot{\phi}_e$-dynamics when x_e changes its sign. This discontinuity will cause difficulties in applying the backstepping technique. To get around this problem, we compute $\dot{\phi}_e$-dynamics based on

$$\begin{aligned}
\sin(\phi_e) &= \frac{x_e\sin(\phi) - y_e\cos(\phi)}{z_e},\\
\cos(\phi_e) &= \frac{x_e\cos(\phi) + y_e\sin(\phi)}{z_e}.
\end{aligned} \tag{2.85}$$

We now use (2.81) and (2.85) to transform (2.81) to

$$\begin{aligned}
\dot{z}_e &= -\cos(\phi_e)\hat{v} + \left(\frac{x_e}{z_e}\frac{\partial x_d}{\partial s} + \frac{y_e}{z_e}\frac{\partial y_d}{\partial s}\right)\dot{s} - \cos(\phi_e)\tilde{v},\\
\dot{\phi}_e &= \hat{w} + \left(\left(\frac{\sin(\phi)}{z_e} - \frac{x_e\sin(\phi_e)}{z_e^2}\right)\frac{\partial x_d}{\partial s} - \left(\frac{\cos(\phi)}{z_e} + \frac{y_e\sin(\phi_e)}{z_e^2}\right)\frac{\partial y_d}{\partial s}\right)\times\\
&\quad \frac{\dot{s}}{\cos(\phi_e)} + \frac{\sin(\phi_e)}{z_e}(\hat{v}+\tilde{v}) + \tilde{w},\\
\dot{\hat{v}} &= \tau_{vc} + \Omega_v,\\
\dot{\hat{w}} &= \tau_{wc} + \Omega_w
\end{aligned} \tag{2.86}$$

It is noted that (2.85) is not defined at $z_e = 0$. However, our controller will guarantee that $z_e(t) \geq z_e^* > 0,\ \forall\, 0 \leq t < \infty$ for feasible initial conditions. The second equation of (2.86) is not defined at $\phi_e(t) = \pm 0.5\pi$ but we will design $\dot{s}$ to overcome this problem. Therefore, we will design the controls τ_{vc} and τ_{wc} for (2.86) to yield the control objective. In the next section, a procedure to design a stabilizer for the path following error system (2.86) is presented in details. The triangular structure of (2.86) suggests us to design the controls τ_{vc} and τ_{wc} in two stages. First, we design the virtual velocity controls for $\hat{v}$ and $\hat{w}$ and choose $\dot{s}$ to ultimately stabilize z_e and ϕ_e at the origin. Based on the backstepping technique, the controls τ_{vc} and τ_{wc} will be then designed.

2.3.2 Control design

Step 1. The z_e and ϕ_e dynamics have three inputs that can be chosen to stabilize z_e and ϕ_e, namely $\dot{s}$, $\hat{v}$ and $\hat{w}$. The input $\hat{w}$ should be designed to stabilize the ϕ_e dynamics at the origin. Therefore, two inputs, $\dot{s}$ and $\hat{v}$, can be used to ultimately stabilize z_e at the origin. We can either choose the input $\hat{v}$ and $\dot{s}$ and then design the remaining input. If we fix $\dot{s}$, then the virtual robot is allowed to move at a desired speed. The real robot will follow the virtual one on the path by the controller, and vice verse. We here choose to fix $\dot{s}$. This allows us to adjust the initial conditions in most cases without moving the robot. Since the transformed system (2.86) is not defined at $z_e = 0$, we first assume in the following that $z_e(t) \geq z_e^* > 0, \forall\, 0 \leq t < \infty$. We will then show that there exist initial conditions such that this hypothesis holds.

Define

$$\begin{aligned} v_e &= \hat{v} - \hat{v}_r, \\ w_e &= \hat{w} - \hat{w}_r \end{aligned} \tag{2.87}$$

where $\hat{v}_r$ and $\hat{w}_r$ are the virtual controls of $\hat{v}$ and $\hat{w}$, respectively. As discussed above, we choose the virtual controls and $\dot{s}$ as follows:

$$\begin{aligned}
\hat{v}_r &= k_1(z_e - \delta_e) + \left(\frac{x_e}{z_e}\frac{\partial x_d}{\partial s} + \frac{y_e}{z_e}\frac{\partial y_d}{\partial s}\right)\frac{v_0(t, z_e)}{\sqrt{\left(\frac{\partial x_d}{\partial s}\right)^2 + \left(\frac{\partial y_d}{\partial s}\right)^2}}, \\
\hat{w}_r &= -k_2\phi_e - \left(\left(\frac{\sin(\phi)}{z_e} - \frac{x_e\sin(\phi_e)}{z_e^2}\right)\frac{\partial x_d}{\partial s} - \left(\frac{\cos(\phi)}{z_e} + \frac{y_e\sin(\phi_e)}{z_e^2}\right)\frac{\partial y_d}{\partial s}\right) \\
&\quad \times \frac{v_0(t, z_e)}{\sqrt{\left(\frac{\partial x_d}{\partial s}\right)^2 + \left(\frac{\partial y_d}{\partial s}\right)^2}} - \frac{\sin(\phi_e)}{z_e}\hat{v}_r - \delta_1\left(\frac{\sin(\phi_e)}{z_e^*}\right)^2\phi_e, \\
\dot{s} &= \frac{\cos(\phi_e)v_0(t, z_e)}{\sqrt{\left(\frac{\partial x_d}{\partial s}\right)^2 + \left(\frac{\partial y_d}{\partial s}\right)^2}}
\end{aligned} \tag{2.88}$$

where $k_1 > 0$, $k_2 > 0$, $\delta_1 > 0$. The term multiplied by δ_1 is a nonlinear damping term to overcome the observer error effect. $v_0(t, z_e) \neq 0$, $\forall t \geq t_0 \geq 0$, $z_e \in \mathbb{R}$, is the speed of the virtual robot on the path. Indeed, one can choose this speed to be a constant. However, the time-varying speed and position path following dependence of the virtual robot on the path is more desirable, especially when the robot starts to follow the path. For example, one might choose

$$v_0(t, z_e) = v_0^*(1 - \chi_1 e^{-\chi_2(t-t_0)})e^{-\chi_3 z_e} \tag{2.89}$$

where $v_0^* \neq 0$, $\chi_i > 0$, $i = 1, 2, 3$, $\chi_1 < 1$. The choice of $v_0(t, z_e)$ in (2.89) has the following desired feature: when the path following error, z_e, is large, the virtual robot will wait for the real one; when z_e is small, the virtual robot will move along the path at the speed closed to v_0^* and the real one follows it within the specified look ahead distance. This feature is suitable in practice because it avoids using a high gain control for large signal z_e. Substituting (2.88) into the first two equations of (2.86) results in

$$\begin{aligned}
\dot{z}_e &= -k_1 \cos(\phi_e)(z_e - \delta_e) - \cos(\phi_e)(\tilde{v} + v_e), \\
\dot{\phi}_e &= -k_2\phi_e - \delta_1 \left(\frac{\sin(\phi_e)}{z_e}\right)^2 \phi_e + \frac{\sin(\phi_e)}{z_e}(\tilde{v} + v_e) + \tilde{w} + w_e.
\end{aligned} \tag{2.90}$$

Step 2. By noting that the virtual control $\hat{v}_r$ is a function of t, x_e, y_e and s, and the virtual control $\hat{w}_r$ is a function of t, x_e, y_e, s and ϕ, differentiating both sides of (2.87) along the solution of the last two equations of (2.86) results in

$$\begin{bmatrix} \dot{v}_e \\ \dot{w}_e \end{bmatrix} = \begin{bmatrix} \tau_{vc} + \Omega_v \\ \tau_{wc} + \Omega_w \end{bmatrix} + \begin{bmatrix} \left(\frac{\partial \hat{v}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{v}_r}{\partial y_e}\sin(\phi)\right)\tilde{v} \\ \left(\frac{\partial \hat{w}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{w}_r}{\partial y_e}\sin(\phi)\right)\tilde{v} - \frac{\partial \hat{w}_r}{\partial \phi}\tilde{w} \end{bmatrix} -$$
$$\begin{bmatrix} \frac{\partial \hat{v}_r}{\partial t} + \frac{\partial \hat{v}_r}{\partial x_e}\left(\frac{\partial x_d}{\partial s}\dot{s} - \hat{v}\cos(\phi)\right) + \frac{\partial \hat{v}_r}{\partial y_e}\left(\frac{\partial y_d}{\partial s}\dot{s} - \hat{v}\sin(\phi)\right) + \frac{\partial \hat{v}_r}{\partial s}\dot{s} \\ \frac{\partial \hat{w}_r}{\partial t} + \frac{\partial \hat{w}_r}{\partial x_e}\left(\frac{\partial x_d}{\partial s}\dot{s} - \hat{v}\cos(\phi)\right) + \frac{\partial \hat{w}_r}{\partial y_e}\left(\frac{\partial y_d}{\partial s}\dot{s} - \hat{v}\sin(\phi)\right) + \frac{\partial \hat{w}_r}{\partial s}\dot{s} + \frac{\partial \hat{w}_r}{\partial \phi}\hat{w} \end{bmatrix} \tag{2.91}$$

From (2.90) and (2.91), we choose the control vector without canceling the useful damping terms and with nonlinear damping terms to overcome the observer error effect as follows:

$$\begin{bmatrix} \tau_{vc} \\ \tau_{wc} \end{bmatrix} = -\begin{bmatrix} c_{21}v_e \\ c_{22}w_e \end{bmatrix} - \delta_2 \begin{bmatrix} \left(\frac{\partial \hat{v}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{v}_r}{\partial y_e}\sin(\phi)\right)^2 v_e \\ \left(\frac{\partial \hat{w}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{w}_r}{\partial y_e}\sin(\phi)\right)^2 w_e + \left(\frac{\partial \hat{w}_r}{\partial \phi}\right)^2 \end{bmatrix} +$$
$$\begin{bmatrix} \frac{\partial \hat{v}_r}{\partial t} + \frac{\partial \hat{v}_r}{\partial x_e}\left(\frac{\partial x_d}{\partial s}\dot{s} - \hat{v}\cos(\phi)\right) + \frac{\partial \hat{v}_r}{\partial y_e}\left(\frac{\partial y_d}{\partial s}\dot{s} - \hat{v}\sin(\phi)\right) + \frac{\partial \hat{v}_r}{\partial s}\dot{s} \\ \frac{\partial \hat{w}_r}{\partial t} + \frac{\partial \hat{w}_r}{\partial x_e}\left(\frac{\partial x_d}{\partial s}\dot{s} - \hat{v}\cos(\phi)\right) + \frac{\partial \hat{w}_r}{\partial y_e}\left(\frac{\partial y_d}{\partial s}\dot{s} - \hat{v}\sin(\phi)\right) + \frac{\partial \hat{w}_r}{\partial s}\dot{s} + \frac{\partial \hat{w}_r}{\partial \phi}\hat{w} \end{bmatrix}$$
$$-\delta_3 \begin{bmatrix} (\hat{v}^2 + \hat{w}^2)v_e \\ (\hat{v}^2 + \hat{w}^2)w_e \end{bmatrix} - \begin{bmatrix} \frac{\sin(\phi_e)}{z_e}\phi_e \\ \phi_e \end{bmatrix} \tag{2.92}$$

where c_{21}, c_{22}, δ_2 and δ_3 are positive constants. The terms multiplied by δ_2 and δ_3 are the nonlinear damping terms to overcome the observer error effect. Substituting (2.92) into (2.91) results in

$$\begin{bmatrix} \dot{v}_e \\ \dot{w}_e \end{bmatrix} = -\begin{bmatrix} c_{21}v_e \\ c_{22}w_e \end{bmatrix} + \begin{bmatrix} \Omega_v \\ \Omega_w \end{bmatrix} - \delta_3 \begin{bmatrix} (\hat{v}^2+\hat{w}^2)v_e \\ (\hat{v}^2+\hat{w}^2)w_e \end{bmatrix} - \begin{bmatrix} \frac{\sin(\phi_e)}{z_e}\phi_e \\ \phi_e \end{bmatrix} +$$
$$\begin{bmatrix} \left(\frac{\partial \hat{v}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{v}_r}{\partial y_e}\sin(\phi)\right)\bar{v} \\ \left(\frac{\partial \hat{w}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{w}_r}{\partial y_e}\sin(\phi)\right)\bar{v} - \frac{\partial \hat{w}_r}{\partial \phi}\tilde{w} \end{bmatrix} -$$
$$\delta_2 \begin{bmatrix} \left(\frac{\partial \hat{v}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{v}_r}{\partial y_e}\sin(\phi)\right)^2 v_e \\ \left(\frac{\partial \hat{w}_r}{\partial x_e}\cos(\phi) + \frac{\partial \hat{w}_r}{\partial y_e}\sin(\phi)\right)^2 w_e + \left(\frac{\partial \hat{w}_r}{\partial \phi}\right)^2 \end{bmatrix}. \tag{2.93}$$

2.3.3 Stability analysis

To analyze the closed loop system consisting of (2.90) and (2.93), we first consider the (ϕ_e, v_e, w_e)-subsystem, then move to the z_e-dynamics.

(ϕ_e, v_e, w_e)-subsystem

For this subsystem, we take the following Lyapunov function

$$V_1 = \frac{1}{2}\left(\phi_e^2 + v_e^2 + w_e^2\right) \tag{2.94}$$

whose the derivative along the solution of the last equation of (2.90) and (2.93), after some manipulation, satisfies

$$\dot{V}_1 \leq -\rho_1 V_1 + \chi_1(\cdot)e^{-\sigma_0(t-t_0)} \tag{2.95}$$

where ρ_1 is a positive constant and can be made arbitrarily large by increasing the design constants k_1, k_2, c_{21} and c_{22}. $\chi_1(\cdot)$ is a class-K function of $\left\|\left(z_e(t_0), \phi_e(t_0), \tilde{\eta}(t_0), \tilde{X}(t_0)\right)\right\|$. From (2.95), it is not hard to show that

$$\|(\phi_e(t), v_e(t), w_e(t))\| \leq \alpha_1(\cdot)e^{-\sigma_1(t-t_0)} \tag{2.96}$$

where $\alpha_1(\cdot)$ is a class-K function of $\left\|\left(z_e(t_0), \phi_e(t_0), \tilde{\eta}(t_0), \tilde{X}(t_0)\right)\right\|$, and σ_1 is a positive constant. Hence (2.96) implies that the (ϕ_e, v_e, w_e)-subsystem is K-exponential stable at the origin.

z_e-dynamics

-Lower-bound of z_e. Defining $\tilde{z}_e = z_e - \delta_e$, the first equation of (2.90) can be written as

$$\begin{aligned}\dot{\tilde{z}}_e &= -k_1\cos(\phi_e)\tilde{z}_e - \cos(\phi_e)(\tilde{v}+v_e)\\ &\geq -k_1\tilde{z}_e - |\tilde{v}+v_e|\\ &\geq -k_1\tilde{z}_e - \alpha_2(\cdot)e^{\sigma_2(t-t_0)}\end{aligned} \tag{2.97}$$

where $\alpha_2(\cdot)$ is a class-K function of $\|(z_e(t_0),\phi_e(t_0),\tilde{\eta}(t_0),\tilde{X}(t_0))\|$, and σ_2 is a positive constant. From (2.97) and comparison principle, we have

$$\tilde{z}_e(t) \geq \tilde{z}_e(t_0)e^{-k_1(t-t_0)} + \frac{\alpha_2(\cdot)}{\sigma_2-k_1}\left(e^{-\sigma_2(t-t_0)} - e^{-k_1(t-t_0)}\right). \tag{2.98}$$

Therefore the condition $z_e(t) \geq z_e^*$ holds if

$$\sigma_2 > k_1,\ z_e(t_0) \geq -2\delta_e + \frac{\alpha_2(\cdot)}{\sigma_2-k_1} + z_e^*. \tag{2.99}$$

-Upper-bound of z_e. To estimate upper-bound of z_e, we write the first equation of (2.90) as

$$\dot{z}_e = -k_1 z_e - k_1(\cos(\phi_e)-1)z_e + k_1\cos(\phi_e)\delta_e - \cos(\phi_e)(\tilde{v}+v_e). \tag{2.100}$$

By taking the Lyapunov function $V_2 = 0.5z_e^2$ and noting that ϕ_e, v_e and $\tilde{v}$ exponentially converge to zero, it is not hard to show that

$$|z_e(t)| \leq \alpha_3(\cdot)e^{-\sigma_3(t-t_0)} + \rho_3 \tag{2.101}$$

where $\alpha_3(\cdot)$ is a class-K function of $\|(z_e(t_0),\phi_e(t_0),\tilde{\eta}(t_0),\tilde{X}(t_0))\|$, and σ and ρ_3 are positive constants. The constant ρ_3 can be made arbitrarily small by reducing δ_e.

2.3.4 Simulations

To illustrate the effectiveness of the proposed output-feedback path following controller, we perform a numerical simulation. The physical parameters are taken from Section 1.3.1, Chapter 1. The observer gains are chosen the same as in Section 2.2.4. The design constants are: $k_1 = 0.5$; $k_2 = 5$; $c_{21} = c_{22} = 2$; $\delta_1 = \delta_2 = \delta_3 = 0.05$; $\delta_e = 0.2$. The initial conditions are: $(\eta^T,\omega^T) = ((-5,0,0.5),(0,1))$, $(\hat{\eta}^T,\hat{X}^T) = ((0,0,0),(0,0))$, $s(0) = 0$. The reference speed of the virtual mobile robot is $v_0 = 5$m / s. The path Γ is chosen to be a sinusoidal path specified by $x_d = s, y_d = 10\sin(0.15s)$. This path is regular and satisfies all requirements in Assumption 2.11. Simulation results are plotted in Figure 2.7. From this figure, it can be clearly seen that the output-feedback proposed controller is able to force the mobile robot in question to follow the reference path nicely.

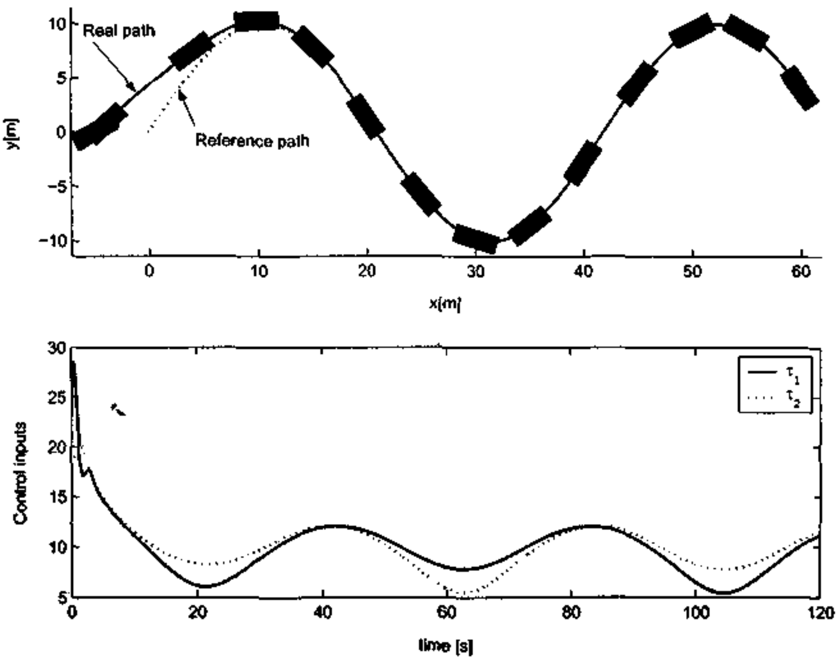


Fig. 2.7. Robot position and orientation in (x, y) plane (top); Control torques (bottom).

2.4 Notes and references

The main difficulty with solving stabilization and tracking control of mobile robots is due to the fact that the motion of the systems in question to be controlled has more degrees of freedom than the number of control inputs under nonholonomic constraints. Brockett's theorem [4] shows that any continuous time invariant feedback control law does not make the null solution of the wheeled mobile robots asymptotically stable. Over the last decade, a lot of interest has been devoted to stabilization and tracking control of nonholonomic mechanical systems including wheeled mobile robots [14], [15], [1], [16], [17] to list a few. Tracking and stabilization are studied separately in these works. In [18], the problem of simultaneous stabilization and tracking was posed and solved for the first time. The control design is based on a special time-varying coordinate transformation, Lyapunov's direct method, and the backstepping technique.

Output-feedback tracking control of land, air, and sea vehicles has been solved for the case of fully actuated [17], pp. 311-334. The main difficulty with designing an observer-based output feedback for Lagrange systems in general is because of the Coriolis matrix, which results in quadratic cross terms of unmeasured velocities. In addition, nonholonomic constraints of mobile robots make the output-feedback problem challenging. For example, many solutions proposed for robot manipulator control ([17] and references therein) cannot directly be applied. Recently, output-feedback tracking of mobile robots was solved in [16]. In this work, based on a special coordinate transformation the exponential observer is designed to estimate the robot velocities. The control design is then based on the time varying coordinate transformation in [18], and the popular backstepping technique [12]. Some other results on output-feedback control of the single-DOF Lagrange systems were addressed in [19] (high-gain control), [11], and [20] for a nonlinear benchmark system. The materials presented in this chapter based on [18], [16], and [21].

Based on the material in this chapter, several control systems developed for stabilization, tracking control and path following of ocean vehicles, which possess second order nonholonomic constraints, and Lemma A.11 are given in [22], [23], [24], [25], [26], [27], [27], [28], [28], [29]. In these papers, both state feedback and output feedback are addressed. A global tracking control solution for a vertical take-off and landing (VTOL) aircraft is given in [30].

It should be also mentioned that all proofs of the main results in this chapter are based on Lyapunov's direct method for the sake of self-containing. Indeed, one can use the stability result for cascade systems given in Appendix A, Section A.1.3 to analyze stability of the closed loop systems. Moreover, if nonlinear damping terms are considered in the robot dynamics, Lemma A.10 can be used with the proposed coordinate transformation to design an exponential/asymptotic observer.

3

Relative Formation Control of Mobile Robots

In this chapter, a constructive method is presented to design cooperative controllers that force a group of N mobile agents to achieve a particular formation in terms of shape and orientation while avoiding collisions between themselves. The control development is based on new local potential functions, which attain the minimum value when the desired formation is achieved, and are equal to infinity when a collision occurs. The proposed controller development is then extended to formation control of a group of unicycle-type mobile robots.

3.1 Departure example

3.1.1 Problem statement

To illustrate our approach, we start with a simple example of a group of two mobile agents. The results for this system will subsequently be extended to the more complicated case with N agents. Consider two mobile agents whose dynamics are given by

$$\dot{q}_i = u_i \tag{3.1}$$

where $q_i = [x_i \ y_i]^T \in \mathbb{R}^2$ and $u_i = [u_{ix} \ u_{iy}]^T \in \mathbb{R}^2$, $i = 1, 2$ are the states and control inputs of each agent. The control objective is to design the controls u_i such that they force the agents to move in formation from initial conditions $q_i(t_0)$, $t_0 \geq 0$ with $\|q_1(t_0) - q_2(t_0)\| > 0$, where $\| \bullet \|$ denotes the standard Euclidian norm of $\bullet$, in the sense that:

1) the agents move in a desired formation: $\lim_{t\to\infty} \|q_1(t) - q_2(t) - l_{12}\| = 0$ where $l_{12} = [l_{x12} \ l_{y12}]^T$ with $\|l_{12}\| > 0$ is desired distance between the agents,

2) no collisions between the agents occur: $\|q_1(t)-q_2(t)\| > 0,\ \forall t \geq t_0 \geq 0$, and

3) the agents' velocity converges to the desired (bounded) velocity $u_d = [u_{dx}\ u_{dy}]^T$: $\lim_{t\to\infty}(u_i(t) - u_d) = 0$.

3.1.2 Control design

Consider the following potential function

$$\varphi = \gamma + \delta\beta \tag{3.2}$$

where δ is a positive tuning constant, γ and β are the goal and related collision avoidance functions, respectively. They are specified below:

-The goal function γ is designed such that it puts penalty on the stabilization error, and is equal to zero when the agents move in the desired formation. A simple choice of this function is

$$\gamma = \frac{1}{2}\|q_1 - q_2 - l_{12}\|^2. \tag{3.3}$$

-The related collision avoidance function β is designed such that it is equal to infinity a collision occurs, and attains the minimum value when the agents move in the desired formation. A possible choice of this function is

$$\beta = \frac{\beta_{12}}{\beta_{12l}^2} + \frac{1}{\beta_{12}} \tag{3.4}$$

where

$$\begin{aligned}\beta_{12} &= 0.5\|q_1 - q_2\|^2,\\ \beta_{12l} &= 0.5\|l_{12}\|^2.\end{aligned} \tag{3.5}$$

To design the controls $u_i = [u_{ix}\ u_{iy}]^T$, differentiating both sides of (3.2) along the solutions of (3.1) gives

$$\begin{aligned}\dot{\varphi} &= \Omega_{12x}(u_{1x} - u_{2x}) + \Omega_{12y}(u_{1y} - u_{2y})\\ &= \Omega_{12x}(u_{1x} - u_{dx} - (u_{2x} - u_{dx})) +\\ &\quad \Omega_{12y}(u_{1y} - u_{dy} - (u_{2y} - u_{dy}))\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\Omega_{12x} &= [x_1 - x_2 - l_{x12}] + \left[\delta\left(\frac{1}{\beta_{12l}^2} - \frac{1}{\beta_{12}^2}\right)(x_1 - x_2)\right],\\ \Omega_{12y} &= [y_1 - y_2 - l_{y12}] + \left[\delta\left(\frac{1}{\beta_{12l}^2} - \frac{1}{\beta_{12}^2}\right)(y_1 - y_2)\right].\end{aligned} \tag{3.7}$$

The equation (3.6) suggests that we choose the controls $u_i = [u_{ix}\ u_{iy}]^T$ as

$$\begin{cases} u_{1x} = -c\Omega_{12x} + u_{dx} \\ u_{1y} = -c\Omega_{12y} + u_{dy} \end{cases} \qquad \begin{cases} u_{2x} = c\Omega_{12x} + u_{dx} \\ u_{2y} = c\Omega_{12y} + u_{dy} \end{cases} \tag{3.8}$$

where c is a positive constant. Substituting (3.8) into (3.6) yields

$$\dot{\varphi} = -2c(\Omega_{12x}^2 + \Omega_{12y}^2). \tag{3.9}$$

Indeed, substituting (3.8) into (3.1) results in the closed loop system

$$\begin{aligned} \dot{q}_1 &= -c\Omega_{12} + u_d \\ \dot{q}_2 &= c\Omega_{12} + u_d \end{aligned} \tag{3.10}$$

where $\Omega_{12} = [\Omega_{12x}\ \Omega_{12y}]^T$.

Remark 3.1. The control pairs (u_{1x}, u_{2x}) and (u_{1y}, u_{2y}) have a special feature in the sense that the first terms (see first square brackets in Ω_{12x} and Ω_{12y}) play the role of driving the agents to their desired locations while the second terms (see second square brackets in Ω_{12x} and Ω_{12y}) take care of collision avoidance between the agents, see (3.7). The second terms act as gyroscopic forces to steer the agents away from each other when they come close to each other. Moreover, the first terms in the above control pairs provide the "attractive forces" that attract the agents to their desired locations. The second terms provide both "attractive and repulsive forces". These forces attract the agents to their desired distance and push the agents away when the agents are too close to each other.

3.1.3 Stability analysis

In this subsection, we show that the controls $u_i = [u_{ix}\ u_{iy}]^T$ given in (3.8) guarantees that no collisions can occur, the solutions of the closed loop system (3.10) exist, the desired formation is asymptotically achieved, and the agents will move with the desired formation velocity u_d.

-Proof of no collisions and existence of solutions.

From (3.9), we have $\dot{\varphi} \leq 0$. Integrating both sides of this inequality gives

$$\gamma(t) + \delta\left(\frac{\beta_{12}(t)}{\beta_{12l}^2} + \frac{1}{\beta_{12}(t)}\right) \leq \gamma(t_0) + \delta\left(\frac{\beta_{12}(t_0)}{\beta_{12l}^2} + \frac{1}{\beta_{12}(t_0)}\right),\ \forall t \geq t_0 \geq 0 \tag{3.11}$$

where

$$\begin{aligned} \gamma(t) &= 0.5\|q_1(t) - q_2(t) - l_{12}\|^2, \\ \gamma(t_0) &= 0.5\|q_1(t_0) - q_2(t_0) - l_{12}\|^2, \\ \beta_{12}(t) &= 0.5\|q_1(t) - q_2(t)\|^2, \\ \beta_{12}(t_0) &= 0.5\|q_1(t_0) - q_2(t_0)\|^2. \end{aligned} \tag{3.12}$$

Since $\|q_1(t_0) - q_2(t_0)\| > 0$ and $\|l_{12}\| > 0$, i.e. $\beta_{12}(t_0) > 0$ and $\beta_{12l} > 0$, the right hand side of (3.11) is bounded. As a result, the left hand side of (3.11) must also be bounded. This means that $\beta_{12}(t) > 0, \forall t \geq t_0 \geq 0$, i.e. no collision between the agents can occur. To show that the solutions of the closed loop system (3.10) exist, we consider the function $V_{12} = 0.5(\|q_1\|^2 + \|q_2\|^2)$ whose derivative along the solutions of the closed loop system (3.10) satisfies $\dot{V}_{12} \leq (a_1/\beta_{12}^2 + a_2)V_{12} + a_3$ where a_1, a_2 and a_3 are some positive constants. This implies that the solutions of the closed loop system (3.10) since $\beta_{12}(t) > 0, \forall t \geq t_0 \geq 0$. Furthermore, applying Barbalat's lemma, see Lemma A.6, to (3.9) gives

$$\lim_{t \to \infty} \Omega_{12} = 0. \tag{3.13}$$

-Behavior near equilibrium points. Since the desired formation is specified in terms of relative distance between the agents, we will consider the dynamics of inter-agents instead of each agent. Defining $q_{12} = q_1 - q_2$ and differentiating this equation along the solutions of the closed loop system (3.10) yield

$$\dot{q}_{12} = -2c\Omega_{12} \tag{3.14}$$

where Ω_{12} is given just below equation (3.10). We can write Ω_{12} as a vector function of $q_{12}$and l_{12} as $\Omega_{12} = q_{12} - l_{12} + \delta\left(1/\beta_{12l}^2 - 1/\beta_{12}^2\right) q_{12}$. At the steady state, we have $\Omega_{12} = 0$. The equation $\Omega_{12} = 0$ has two roots $q_{12} = l_{12}$ and $q_{12} = q_{12c}$, $q_{12c} = [x_{12c}\ y_{12c}]^T$. Therefore (3.13) implies that q_{12} approaches either l_{12} or q_{12c}.

It is noted that q_{12c} has a property that the term $q_{12c}^T l_{12}$ is strictly negative, i.e. the point at which $(x_{12}, y_{12}) = (0, 0)$ locates between the equilibrium point (x_{12c}, y_{12c}) and the equilibrium point (l_{x12}, l_{y12}). This is due to the fact that at the equilibrium point (l_{x12}, l_{y12}) all the attractive and repulsive forces are equal to zero while at the critical point (x_{12c}, y_{12c}) the sum of attractive and repulsive forces (but they are different from zero) is equal to zero. This can be viewed graphically in Figure 3.1.

In the rest of this section, we will show that the equilibrium point (l_{x12}, l_{y12}) is asymptotically stable while the equilibrium point (x_{12c}, y_{12c}) is saddle. The general gradient of $\Omega_{12}(q_{12}, l_{12})$ with respect to q_{12} is given by

$$\frac{\partial \Omega_{12}}{\partial q_{12}} = \begin{bmatrix} 1 + \delta\left(\frac{1}{\beta_{12l}^2} - \frac{1}{\beta_{12}^2}\right) + 2\delta\frac{x_{12}^2}{\beta_{12}^3} & \frac{2\delta x_{12} y_{12}}{\beta_{12}^3} \\ \frac{2\delta x_{12} y_{12}}{\beta_{12}^3} & 1 + \delta\left(\frac{1}{\beta_{12l}^2} - \frac{1}{\beta_{12}^2}\right) + \frac{2\delta y_{12}^2}{\beta_{12}^3} \end{bmatrix}. \tag{3.15}$$

To show that the equilibrium point (l_{x12}, l_{y12}) is asymptotically stable, we need to show that the matrix $A_{l_{12}} = \left.\frac{\partial \Omega_{12}}{\partial q_{12}}\right|_{q_{12}=l_{12}}$ is positive definite. Substituting $q_{12} = l_{12}$ into (3.15) yields

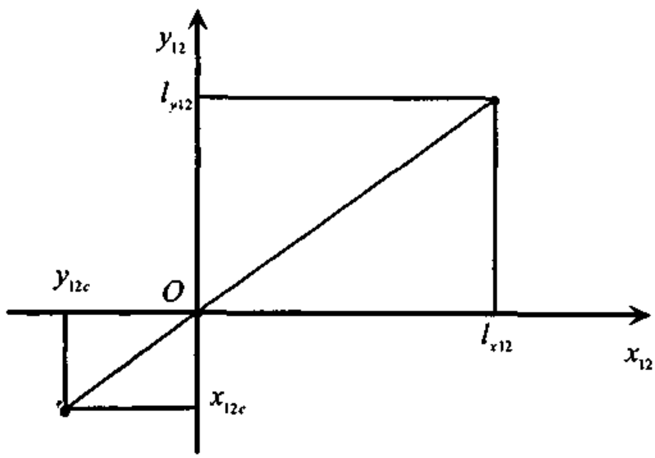


Fig. 3.1. Illustrating location of equilibrium points.

$$A_{l_{12}} = \begin{bmatrix} 1 + \frac{2\delta l_{x12}^2}{\beta_{12l}^3} & \frac{2\delta l_{x12} l_{y12}}{\beta_{12l}^3} \\ \frac{2\delta l_{x12} l_{y12}}{\beta_{12l}^3} & 1 + \frac{2\delta l_{y12}^2}{\beta_{12l}^3} \end{bmatrix}. \tag{3.16}$$

Since $1 + 2\delta l_{x12}^2/\beta_{12l}^3 > 0$ and $\det(A_{l_{12}}) = 1 + 2\delta/\beta_{12l}^2 > 0$ where $\det(\bullet)$ denotes the determinant of $\bullet$, the matrix $A_{l_{12}}$ is positive definite, i.e. the equilibrium point (l_{x12}, l_{y12}) is asymptotically stable. On the other hand at the equilibrium point (x_{12c}, y_{12c}), we have

$$\left.\frac{\partial \Omega_{12}}{\partial q_{12}}\right|_{q_{12}=q_{12c}} = \begin{bmatrix} 1 + \delta\left(\frac{1}{\beta_{12l}^2} - \frac{1}{\beta_{12c}^2}\right) + \frac{2\delta x_{12c}^2}{\beta_{12c}^3} & \frac{2\delta x_{12c} y_{12c}}{\beta_{12c}^3} \\ \frac{2\delta x_{12c} y_{12c}}{\beta_{12c}^3} & 1 + \delta\left(\frac{1}{\beta_{12l}^2} - \frac{1}{\beta_{12c}^2}\right) + \frac{2\delta y_{12c}^2}{\beta_{12c}^3} \end{bmatrix} \triangleq A_{q_{12c}} \tag{3.17}$$

where $\beta_{12c} = 0.5\|q_{12c}\|^2$. The determinant of the matrix $A_{q_{12c}}$ is given by

$$\det(A_{q_{12c}}) = \left(1 + \delta\left(\frac{1}{\beta_{12l}^2} - \frac{1}{\beta_{12c}^2}\right)\right)\left(1 + \delta\left(\frac{1}{\beta_{12l}^2} + \frac{3}{\beta_{12c}^2}\right)\right). \tag{3.18}$$

Since at the equilibrium point (x_{12c}, y_{12c}), we have $\Omega_{12c} = 0$ where Ω_{12c} is Ω_{12} being evaluated at $q_{12} = q_{12c}$. Multiplying both sides of $\Omega_{12c} = 0$ with q_{12c}^T, we have $q_{12c}^T \Omega_{12c} = 0$. Expanding $q_{12c}^T \Omega_{12c} = 0$ gives

$$\delta\left(1/\beta_{12l}^2 - 1/\beta_{12c}^2\right) = -\frac{1}{2\beta_{12c}} q_{12c}^T (q_{12c} - l_{12}). \tag{3.19}$$

Substituting (3.19) into (3.18) yields

$$\det(A_{q_{12c}}) = \frac{q_{12c}^T l_{12}}{2\beta_{12c}} \left(1 + \delta \left(1/\beta_{12l}^2 + 3/\beta_{12c}^2\right)\right). \quad (3.20)$$

Since $q_{12c}^T l_{12}$ is strictly negative, we have $\det(A_{q_{12c}}) < 0$, which implies that the equilibrium point (x_{12c}, y_{12c}) is saddle.

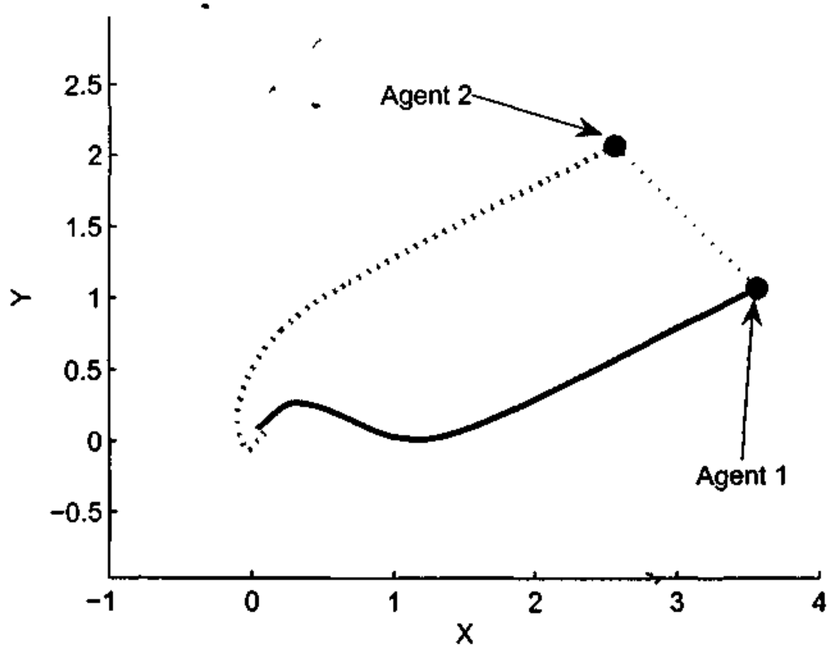


Fig. 3.2. Agents' motion in (x, y) plane.

3.1.4 Simulations

We now illustrate the result of the previous subsection by running a simulation with $l_{12} = [1\ \ -1]^T$, $c = 1$, $\delta = 0.2$, $u_d = [1\ 0.5]^T$. For this set of numerical values, the equilibrium points of the agents' inter-dynamics are $(l_{x12}, l_{y12}) = (1, -1)$ and $(x_{12c}, y_{12c}) = (-0.5, 0.5)$. Indeed, it is true that $q_{12c}^T l_{12} = -1$ which is strictly negative. The agents are initialized randomly in a circle with radius of 0.1 centered at the origin. The agents' motions in the (x, y) plane are

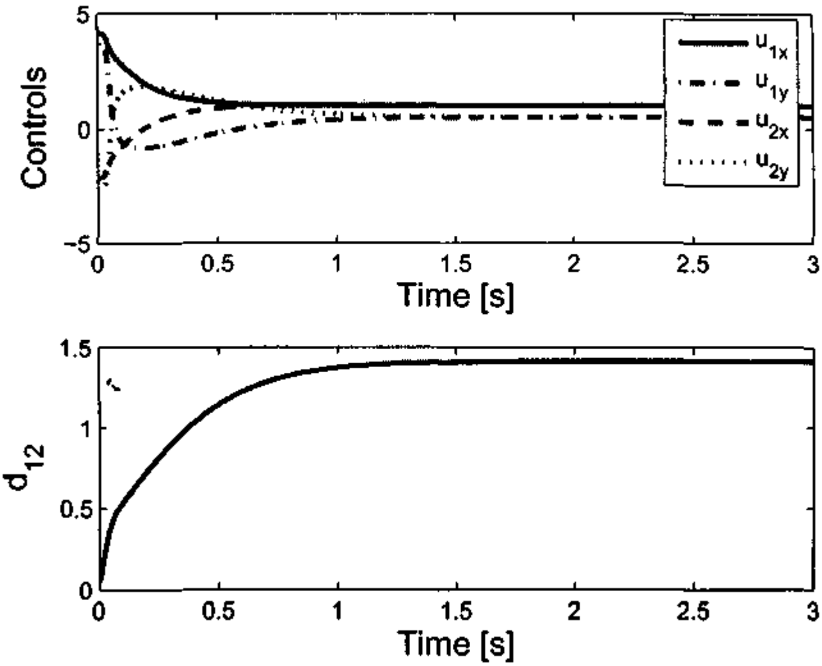


Fig. 3.3. Controls and distance between agents.

plotted in Figure 3.2. Figure 3.3 plots the controls and distance between the agents, $d_{12} = ||q_1 - q_2||$. Clearly, there are no collisions since d_{12} always larger than zero. In addition, the agents' velocities converge to the desired velocity u_d. The phase portraits of the inter-agent dynamics are plotted in Figure 3.4. It is observed from Figure 3.4 that the goal point (l_{x12}, l_{y12}) is the attractive point (arrows go to this point) and that at this point all the attractive and repulsive forces are zero. The zero point $(0, 0)$ is repulsive point (arrows go out from this point). At the saddle point (x_{12c}, y_{12c}), sum of attractive and repulsive forces is equal to zero (some arrows go in and some arrows go out).

3.2 Formation control of N agents

In this section, we generalize the results for the simple system in the previous section to a more complex system of N mobile agents.

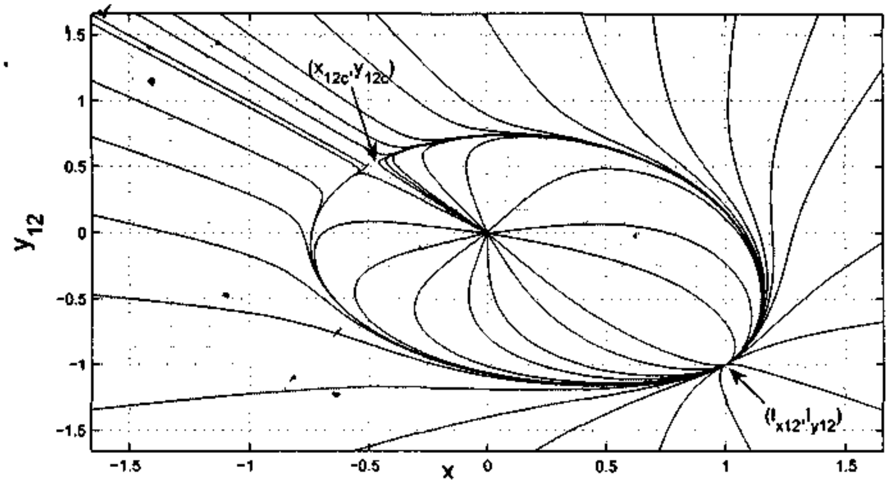


Fig. 3.4. Phase portraits of inter-agent dynamics.

3.2.1 Problem statement

We consider a group of N simple point-mass mobile robots, of which each has the following dynamics

$$\dot{q}_i = u_i, \quad i = 1, ..., N \tag{3.21}$$

where $q_i \in \mathrm{R}^n$ and $u_i \in \mathrm{R}^n$ are the state and control input of the robot i. We assume that $n > 1$ and $N > 1$. The assumption that each robot is represented as a point is not as restrictive as it may seem since various shapes can be mapped to single points through a series of transformations [31], [32], [33]. Our task is to design the control input u_i for each robot i that forces the group of N robots to stabilize with respect to their group members in configurations that make a particular formation specified by a desired vector $l(\eta) = [l_{12}^T(\eta), l_{23}^T(\eta), ..., l_{N-1,N}^T(\eta)]^T$, where $\eta \in \mathrm{R}^m$ is the formation parameter vector to specify the formation change, while avoiding collisions between themselves. The parameter vector η is used to specify rotation, expansion and contraction of the formation such that when η converges to its desired value η_f, the desired shape of the formation is achieved. In addition, it requires all the robots align their velocity vectors to a desired bounded one $u_d \in \mathrm{R}^n$, and move toward specified directions specified by the desired formation velocity vector. The control objective is formally stated as follows:

Control objective: Assume that at the initial time t_0 each robot initializes at a different location, and that each robot has a different desired location, i.e. there exist strictly positive constants ε_1, ε_2 and ε_3 such that

$$\begin{array}{l} \|q_i(t_0) - q_j(t_0)\| \geq \varepsilon_1, \\ \|l_{ij}(\eta)\| \geq \varepsilon_2, \\ \|\partial l_{ij}(\eta)/\partial\eta\| \leq \varepsilon_3, \ \forall i, j \in \{1, 2, ...N\}, \ \forall \eta \in \mathrm{R}^m. \end{array} \tag{3.22}$$

Design the control input u_i for each robot i , and an update law for the formation parameter vector η such that each robot (almost) globally asymptotically approaches its desired location to form a desired formation, and that the robots' velocity converges to the desired (bounded) velocity u_d while avoiding collisions with all other robots in the group, i.e.

$$\begin{array}{l} \lim_{t\to\infty}(q_i(t) - q_j(t) - l_{ij}(\eta(t))) = 0, \\ \lim_{t\to\infty}(\eta(t) - \eta_f) = 0, \\ \lim_{t\to\infty}(u_i(t) - u_d) = 0, \\ \|q_i(t) - q_j(t)\| > \varepsilon_4, \ \forall i, j \in \{1, 2, ...N\}, \ \forall t \geq t_0 \geq 0 \end{array} \tag{3.23}$$

where ε_4 is a strictly positive constant, and η_f is a vector of constants that determine the desired formation. The desired formation can be represented by a labeled directed graph ([34], [35]) in the following definition.

Definition 3.2. *The formation graph, $G = \{V, E, L\}$ is a directed labeled graph consisting of:*

-a set of vertices (nodes), $V = \{\vartheta_1, \cdots, \vartheta_N\}$ indexed by the mobile robots in the group,

-a set of edges, $E = \{(\vartheta_i, \vartheta_j) \in V \times V\}$, containing ordered pairs of vertices that represent inter-robot position constraints, and

-a set of labels, $L = \{\gamma_{ij} | \gamma_{ij} = \|q_i - q_j - l_{ij}\|^2, \ \forall(\vartheta_i, \vartheta_j) \in E\}$, $l_{ij} = q_{if} - q_{jf} \in \mathrm{R}^n$ indexed by the edges in E.

Indeed, when the control objective is achieved, the edge labels become $\|q_i - q_j - l_{ij}\|^2 = 0$, $\forall(\vartheta_i, \vartheta_j) \in E$, i.e. the relative distance between the robots i and j is l_{ij}.

3.2.2 Control design

We consider the following local potential function

$$\varphi_i = \gamma_i + \delta\beta_i \tag{3.24}$$

where δ are positive tuning constants, the functions γ_i and β_i are the goal and related collision avoidance functions for the robot i specified as follows:

-The goal function γ_i is essentially the sum of all distances from the robot i to its adjacent group members, N_i. A simple choice of this function is

$$\gamma_i = \sum_{j \in N_i} \gamma_{ij}, \qquad \gamma_{ij} = \frac{1}{2}\|q_i - q_j - l_{ij}\|^2. \tag{3.25}$$

-The related collision function β_i should be chosen such that it is equal to infinity whenever any robots come in contact with the robot i, i.e. a collision occurs, and attains the minimum value when the robot i is at its desired location with respect to other group members belong to N_i, which are adjacent to the robot i. This function is chosen as follows:

$$\beta_i = \sum_{j \in N_i} \left(\frac{\beta_{ij}^k}{\beta_{ijl}^{2k}} + \frac{1}{\beta_{ij}^k} \right) \tag{3.26}$$

where k is a positive constant to be chosen later, β_{ij} and β_{ijl} are collision and desired collision functions chosen as

$$\beta_{ij} = \frac{1}{2}||q_i - q_j||^2, \quad \beta_{ijl} = \frac{1}{2}||l_{ij}||^2. \tag{3.27}$$

It is noted from (3.27) that $\beta_{ij} = \beta_{ji}$ and $\beta_{ijl} = \beta_{jil}$.

Remark 3.3. 1) The above choice of the potential function φ_i given in (3.24) with its components specified in (3.25)-(3.26), has the following properties: 1) it attains the minimum value when the robot i is at the desired location with respect to other group member belong to N_i, which are adjacent to the robot i, i.e. $q_i - q_j - l_{ij} = 0$, $j \in N_i$, and 2) it is equal to infinity whenever one or more robots come in contact with the robot i, i.e. when a collision occurs.

2) The potential function (3.24) is different from the ones proposed in [36] and [37] in the sense that the ones in [36] and [37] are centralized and do not put penalty on the relative distance between the robots, i.e. do not include the goal function γ_i. Therefore, the controllers developed in [36] and [37] do not guarantee the formation converge to a specified configuration but to any configurations that locally minimize the potential functions (these potential functions in [36] and [37] are nonconvex).

3) Our potential function (3.24) is also different from the navigation functions proposed in [31] and [34] in the sense that our potential function is of the form of sum of collision avoidance functions while those navigation functions in [31] and [34] are of the form of product of collision avoidance functions . This feature makes our potential function "more decentralized". Furthermore, our potential function is equal to infinity while those in [36], [31] and [34] is equal to a finite constant when a collision occurs. However, those in [31] and [34] also cover obstacle and work space boundary avoidance. Although these issues are not included in this paper for clarity, considering these issues is possible and is the subject of future work.

4) Our potential function does not have problems like local minima and non-reachable goal as listed in [38].

To design the control input u_i, we differentiate both sides of (3.24) along the solutions of (3.21) to obtain

$$\begin{aligned}\dot{\varphi}_i &= \sum_{j\in N_i}[\Omega_{ij}^T(u_i-u_j)-\Psi_{ij}^T\dot{\eta}] \\ &= \sum_{j\in N_i}[\Omega_{ij}^T(u_i-u_d-(u_j-u_d))-\Psi_{ij}^T\dot{\eta}] \\ &= \sum_{j\in N_i}\Omega_{ij}^T(u_i-u_d)-\sum_{j\in N_i}\Omega_{ij}^T(u_j-u_d)-\sum_{j\in N_i}\Psi_{ij}^T\dot{\eta}\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}\Omega_{ij} &= q_i-q_j-l_{ij}+\delta k\left(\frac{1}{\beta_{ijl}^{2k}}-\frac{1}{\beta_{ij}^{2k}}\right)\beta_{ij}^{k-1}(q_i-q_j) \\ \Psi_{ij} &= \left[\left(q_i-q_j-l_{ij}+\frac{2\delta k\beta_{ij}^k}{\beta_{ijl}^{2k+1}}l_{ij}\right)^T\frac{\partial l_{ij}}{\partial\eta}\right]^T.\end{aligned} \tag{3.29}$$

From (3.28), we simply choose the control u_i for the robot i and the update law for η as follows:

$$\begin{aligned}u_i &= -C\sum_{j\in N_i}\Omega_{ij}+u_d \\ \dot{\eta} &= -\Gamma(\eta-\eta_f)\end{aligned} \tag{3.30}$$

where $C\in\mathbb{R}_+^{n\times n}$ and $\Gamma\in\mathbb{R}_+^{m\times m}$ are symmetric positive definite matrices. Substituting (3.30) into (3.28) yields

$$\dot{\varphi}_i = -\sum_{j\in N_i}\Omega_{ij}^TC\sum_{j\in N_i}\Omega_{ij}-\sum_{j\in N_i}\Omega_{ij}^T(u_j-u_d)+\sum_{j\in N_i}\Psi_{ij}^T\Gamma(\eta-\eta_f). \tag{3.31}$$

Substituting (3.30) into (3.21) results in the closed loop system

$$\dot{q}_i = -C\sum_{j\in N_i}\Omega_{ij}+u_d,\ i=1,...,N. \tag{3.32}$$

Since the desired formation is specified in terms on relative distances between the robots, we write the closed loop system of the inter-robot dynamics from the closed loop system (3.32) as

$$\dot{q}_{ij} = -C\left(\sum_{a\in N_i}\Omega_{ia}-\sum_{b\in N_j}\Omega_{jb}\right),\ (i,j)\in\{1,...,N\},\ i\neq j \tag{3.33}$$

where $q_{ij}=q_i-q_j$. We now state the main result in the following theorem.

Theorem 3.4 *Under the assumptions stated in the control objective, the control for each robot i given in (3.30) with an appropriate choice of the tuning constants δ and k, solves the control objective.*

3.2.3 Proof of Theorem 3.4

We prove Theorem 3.4 in two steps. At the first step, we show that there are no collisions between any robots and the solutions of the closed loop system exist. At the second step, we prove that the equilibrium point of the inter-robot dynamics closed loop system (3.33), at which $q_i - q_j - l_{ij} = 0$, is asymptotically stable. Finally, we show that all other equilibrium(s) of (3.33) are either unstable or saddle.

Step 1: Proof of no collision and existence of solutions
We consider the following common potential function φ given by

$$\varphi = \sum_{i=1}^{N} \varphi_i \tag{3.34}$$

whose derivative along the solutions of (3.31) is

$$\dot{\varphi} = -\sum_{i=1}^{N} \sum_{j \in N_i} \Omega_{ij}^T C \sum_{j \in N_i} \Omega_{ij} - \sum_{i=1}^{N} \sum_{j \in N_i} \Omega_{ij}^T (u_j - u_d) + \sum_{i=1}^{N} \sum_{j \in N_i} \Psi_{ij}^T \Gamma (\eta - \eta_f). \tag{3.35}$$

Since $l_{ij} = -l_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$, we have

$$\sum_{i=1}^{N} \sum_{j \in N_i} \Omega_{ij}^T (u_j - u_d) = -\sum_{i=1}^{N} \sum_{j \in N_i} \Omega_{ij}^T (u_i - u_d). \tag{3.36}$$

Substituting (3.36) into (3.35) gives

$$\dot{\varphi} = -2\sum_{i=1}^{N} \sum_{j \in N_i} \Omega_{ij}^T C \sum_{j \in N_i} \Omega_{ij} + \sum_{i=1}^{N} \sum_{j \in N_i} \Psi_{ij}^T \Gamma (\eta - \eta_f). \tag{3.37}$$

We now consider the following total function $\varphi_{tot} = \log(1 + \varphi) + 0.5\|\eta - \eta_f\|^2$ whose derivative along the solutions of (3.37) the second equation of (3.30) satisfies

$$\begin{aligned} \dot{\varphi}_{tot} = & -\frac{2}{1+\varphi} \sum_{i=1}^{N} \sum_{j \in N_i} \Omega_{ij}^T C \sum_{j \in N_i} \Omega_{ij} + \\ & \frac{1}{1+\varphi} \sum_{i=1}^{N} \sum_{j \in N_i} \Psi_{ij}^T \Gamma (\eta - \eta_f) - (\eta - \eta_f)^T \Gamma (\eta - \eta_f) \end{aligned} \tag{3.38}$$

which implies that

$$\dot{\varphi}_{tot} \leq -\frac{2}{1+\varphi}\sum_{i=1}^{N}\sum_{j\in N_i}\Omega_{ij}^T C \sum_{j\in N_i}\Omega_{ij} + \frac{\lambda_{\max}(\Gamma)}{4\varepsilon(1+\varphi)^2}\left\|\sum_{i=1}^{N}\sum_{j\in N_i}\Psi_{ij}^T\right\|^2 - (\lambda_{\min}(\Gamma) - \varepsilon\lambda_{\max}(\Gamma))\|\eta - \eta_f\|^2 \tag{3.39}$$

where ε is a positive constant, $\lambda_{\min}(\Gamma)$ and $\lambda_{\max}(\Gamma)$ denote the minimum and maximum eigenvalues of Γ respectively. From (3.29), and definition of the function φ, it can be readily shown that there exists a positive constant $\omega_{\max}$ such that

$$\frac{1}{(1+\varphi)^2}\left\|\sum_{i=1}^{N}\sum_{j\in N_i}\Psi_{ij}^T\right\|^2 \leq \omega_{\max} \tag{3.40}$$

With (3.40) in mind, picking $\varepsilon = \lambda_{\min}(\Gamma)/\lambda_{\max}(\Gamma)$ we can write (3.39) as

$$\dot{\varphi}_{tot} \leq \frac{\lambda_{\max}^2(\Gamma)}{4\lambda_{\min}(\Gamma)}\omega_{\max} \triangleq \varpi_{\max}. \tag{3.41}$$

Integrating both sides of (3.41) results in

$$\varphi_{tot}(t) \leq \varphi_{tot}(t_0) + \varpi_{\max}(t - t_0). \tag{3.42}$$

where $\varphi_{tot}(t)$ and $\varphi_{tot}(t_0)$ are (from the definition of φ_{tot})

$$\begin{aligned}\varphi_{tot}(t) &= \log\left[1 + \sum_{i=1}^{N}\left(\gamma_i(t) + \delta\sum_{j\in N_i}\left(\frac{\beta_{ij}^k(t)}{\beta_{ijl}^{2k}} + \frac{1}{\beta_{ij}^k(t)}\right)\right)\right] + \tfrac{1}{2}\|\eta(t) - \eta_f\|^2 \\ \varphi_{tot}(t_0) &= \log\left[1 + \sum_{i=1}^{N}\left(\gamma_i(t_0) + \delta\sum_{j\in N_i}\left(\frac{\beta_{ij}^k(t_0)}{\beta_{ijl}^{2k}} + \frac{1}{\beta_{ij}^k(t_0)}\right)\right)\right] + \\ &\quad \tfrac{1}{2}\|\eta(t_0) - \eta_f\|^2.\end{aligned} \tag{3.43}$$

The right hand side of (3.42) cannot escape to infinity unless when $t = \infty$ since $\beta_{ijl} > 0$ and $\beta_{ij}(t_0) > 0$ (see definition of β_{ijl} and β_{ij} given in (3.27)). Therefore the left hand side of (3.42) cannot escape to infinity for all $t \in [t_0, \infty)$. This implies that $\beta_{ij}(t)$ cannot be zero for all $t \in [t_0, \infty)$, i.e. no collisions can occur for all $t \in [t_0, \infty)$. On the other hand, it is true from the second equation of (3.30) that

$$\|\eta(t) - \eta_f\| \leq \|\eta(t_0) - \eta_f\| e^{-\lambda_{\min}(\Gamma)(t-t_0)} \tag{3.44}$$

which means that the desired formation shape is achieved exponentially. Using (3.40) and (3.44), we can write (3.39) as

$$\dot{\varphi}_{tot} \leq \lambda_{\max}(\Gamma)\sqrt{\omega_{\max}}\|\eta(t_0) - \eta_f\| e^{-\lambda_{\min}(\Gamma)(t-t_0)}. \tag{3.45}$$

Integrating both sides of (3.45) from t_0 to t results in

$$\varphi_{tot}(t) \leq \varphi_{tot}(t_0) + \lambda_{\max}(\Gamma)\sqrt{\omega_{\max}}\|\eta(t_0) - \eta_f\|/\lambda_{\min}(\Gamma). \tag{3.46}$$

It is seen that the right hand side of (3.46) is bounded. Therefore the left hand side of (3.46) must also be bounded. This implies that $\beta_{ij}(t)$ must be larger than a strictly positive constant for all $t \in [t_0, \infty)$, which in turn means that there exists a strictly positive constant ε_4 such that the last inequality of (3.23) holds. To prove that the solutions of the closed loop system (3.32) exist, we consider the function $W = 0.5 \sum_{i=1}^{N} \|q_i\|^2$ whose derivative along the solutions of (3.32), after some simple manipulation, satisfies $\dot{W} \leq \rho_1(1 + 1/\min(\beta_{ij}))W + \rho_2$, where ρ_1 and ρ_2 are some positive constants, which implies that the solutions of (3.32) exist since $\beta_{ij}(t)$ is larger than a strictly positive constant for all $t \in [t_0, \infty)$. Furthermore, applying Barbalat's lemma, see Lemma A.6, to (3.39) gives

$$\lim_{t\to\infty} \frac{1}{1+\varphi(t)} \sum_{i=1}^{N} \sum_{j\in N_i} \Omega_{ij}^T(t) C \sum_{j\in N_i} \Omega_{ij}(t) = 0 \tag{3.47}$$

which implies that

$$\begin{cases} \lim_{t\to\infty} \sum_{j\in N_i} \Omega_{ij}(t) = 0 \\ \lim_{t\to\infty} \varphi(t) = \chi_1 \end{cases} \quad \text{or} \quad \begin{cases} \lim_{t\to\infty} \sum_{j\in N_i} \Omega_{ij}(t) = \chi_2 \\ \lim_{t\to\infty} \varphi(t) = \infty \end{cases} \tag{3.48}$$

where χ_1 and χ_2 are some constants. From definitions of Ω_{ij} and φ, the second limit set in (3.48) cannot be true. Therefore, the first limit set in (3.48) implies that $\lim_{t\to\infty} \sum_{j\in N_i} \Omega_{ij}(t) = 0$.

Step 2: Behavior near equilibrium points

At the steady state, the equilibrium points are found by solving the following equations

$$\sum_{j\in N_i} \Omega_{ij} = \sum_{j\in N_i} \left(q_{ij} - l_{ij} + \delta k \left(\frac{1}{\beta_{ijl}^{2k}} - \frac{1}{\beta_{ij}^{2k}} \right) \beta_{ij}^{k-1} q_{ij} \right) = 0 \tag{3.49}$$

for all $i = 1, ..., N$. It is directly verified that $\bar{q} = \bar{l}$ where $\bar{q}$ and are stack vectors of q_{ij} and l_{ij}, respectively, i.e. $\bar{q} = [q_{12}^T, q_{13}^T ..., q_{N-1,N}^T]^T$ and $\bar{l} = [l_{12}^T, l_{13}^T ..., l_{N-1,N}^T]^T$, is one root of (3.49). In addition there is (are) another root(s) denoted by $\bar{q}_c = [q_{12c}^T, q_{13c}^T ..., q_{N-1,Nc}^T]^T$ of (3.49) different from $\bar{l}$ satisfying

$$\sum_{j\in N_i} \Omega_{ij} \bigg|_{\bar{q}=\bar{q}_c} = \sum_{j\in N_i} \left(q_{ijc} - l_{ij} + \delta k \left(\frac{1}{\beta_{ijl}^{2k}} - \frac{1}{\beta_{ijc}^{2k}} \right) \beta_{ijc}^{k-1} q_{ijc} \right) = 0 \tag{3.50}$$

for all $i = 1, ..., N$, where $\beta_{ijc} = 0.5||q_{ic} - q_{jc}||^2$. In the following, we will show that the equilibrium point $\bar{q} = \bar{l}$ is asymptotically stable, and the equilibrium point(s) $\bar{q} = \bar{q}_c$ is (are) unstable or saddle. We now write the closed loop system of the inter-robot dynamics (3.33) as

$$\dot{\bar{q}} = -\bar{C}F(\bar{q}, \bar{l}). \tag{3.51}$$

where $\bar{C} = \text{diag}(\underbrace{C, \cdots, C}_{E})$ with E the number of edges of the formation graph, and

$$\begin{aligned} F(\bar{q}, \bar{l}) = & \Big[\sum_{a \in N_1} \Omega_{1a}^T - \sum_{b \in N_2} \Omega_{2b}^T, \sum_{a \in N_1} \Omega_{1a}^T - \sum_{b \in N_3} \Omega_{3b}^T, ..., \\ & \sum_{a \in N_i} \Omega_{ia}^T - \sum_{b \in N_j} \Omega_{jb}^T, ..., \sum_{a \in N_{N-1}} \Omega_{N-1,a}^T - \sum_{b \in N_N} \Omega_{Nb}^T \Big]^T. \end{aligned} \tag{3.52}$$

Since (3.50) holds for all $i = 1, ..., N$, at the steady state we have $\sum_{a \in N_i} \Omega_{ia} - \sum_{b \in N_j} \Omega_{jb} = 0, \ \forall (i,j) \in \{1, ..., N\}, \ i \neq j$. Therefore the equilibrium points $\bar{q} = \bar{l}$ and $\bar{q} = \bar{q}_c$ are also the equilibrium points of (3.51). The general gradient of $F(\bar{q}, \bar{l})$ with respect to $\bar{q}$ is given by

$$\begin{aligned} \frac{\partial F(\bar{q}, \bar{l})}{\partial \bar{q}} &= \begin{bmatrix} \frac{\partial \Xi_{12}}{\partial q_{12}} & \frac{\partial \Xi_{12}}{\partial q_{13}} & \cdots & \frac{\partial \Xi_{12}}{\partial q_{N-1,N}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Xi_{ij}}{\partial q_{12}} & \cdots & \frac{\partial \Xi_{ij}}{\partial q_{ij}} & \frac{\partial \Xi_{ij}}{\partial q_{N-1,N}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Xi_{N-1,N}}{\partial q_{12}} & \cdots & \cdots & \frac{\partial \Xi_{N-1,N}}{\partial q_{N-1,N}} \end{bmatrix}, \\ \Xi_{ij} &= \sum_{a \in N_i} \Omega_{ia} - \sum_{b \in N_j} \Omega_{jb}, \ (i,j) \in \{1, ..., N\}, \ i \neq j. \end{aligned} \tag{3.53}$$

It can be checked that

$$\begin{aligned} \frac{\partial \Xi_{ij}}{\partial q_{ij}} &= NI_{n\times n} + 2\delta k \left(\frac{1}{\beta_{ijl}^{2k}} - \frac{1}{\beta_{ij}^{2k}}\right) \beta_{ij}^{k-1} I_{n\times n} + \\ & \quad 2\delta k \left((k-1)\left(\frac{1}{\beta_{ijl}^{2k}} - \frac{1}{\beta_{ij}^{2k}}\right)\beta_{ij}^{k-2} + \frac{2k}{\beta_{ij}^{k+2}}\right) q_{ij} q_{ij}^T \triangleq H_{ij} \\ \frac{\partial \Xi_{ij}}{\partial q_{cd}} &= s\delta k \left(\frac{1}{\beta_{cdl}^{2k}} - \frac{1}{\beta_{cd}^{2k}}\right) \beta_{cd}^{k-1} I_{n\times n} + \\ & \quad s\delta k \left((k-1)\left(\frac{1}{\beta_{cdl}^{2k}} - \frac{1}{\beta_{cd}^{2k}}\right)\beta_{cd}^{k-2} + \frac{2k}{\beta_{cd}^{k+2}}\right) q_{cd} q_{cd}^T, \end{aligned} \tag{3.54}$$

where $(c, d) \in \{1, ..., N\}$, $(c, d) \neq (i, j)$, $c \neq d$, and $s = 1$ or $s = -1$ depending on value of c, d, i and j. However, we do not need to specify the sign of

s for our next task. We now investigate properties of the equilibrium points $\bar{q} = \bar{l}$ and $\bar{q} = \bar{q}_c$ based on the general gradient $\partial F(\bar{q}, \bar{l})/\partial \bar{q}$ evaluated at those points.

Step 2.1 Proof of $\bar{q} = \bar{l}$ being the asymptotic stable equilibrium point
At the equilibrium point $\bar{q} = \bar{l}$, we have

$$\left.\frac{\partial \Xi_{ij}}{\partial q_{ij}}\right|_{\bar{q}=\bar{l}} = NI_{n\times n} + \frac{4\delta k^2}{\beta_{ijl}^{k+2}} l_{ij} l_{ij}^T, \quad \left.\frac{\partial \Xi_{ij}}{\partial q_{cd}}\right|_{\bar{q}=\bar{l}} = s\frac{2\delta k^2}{\beta_{cdl}^{k+2}} l_{cd} l_{cd}^T, \tag{3.55}$$

where $\beta_{cdl} = 0.5\|l_{cd}\|^2$. With (3.55), let $\xi \in \mathbb{R}^{nE}$ we have

$$\xi^T \left.\frac{\partial F(\bar{q}, \bar{l})}{\partial \bar{q}}\right|_{\bar{q}=\bar{l}} \xi \geq \left(N - \frac{4\delta k^2 nE \max(l_{ija}^2)}{\min(\beta_{ijl}^{k+2})}\right) \xi^T \xi, \quad (i,j) \in \{1,...,N\},\ i \neq j \tag{3.56}$$

where l_{ija} is the a^{th} element of l_{ij}. Therefore, for any given constant k if we choose the tuning constant δ such that

$$N - \frac{4\delta k^2 nE \max(l_{ija}^2)}{\min(\beta_{ijl}^{k+2})} > 0$$
$$\rightarrow\ \delta < \frac{N \min(\beta_{ijl}^{k+2})}{4k^2 nE \max(l_{ija}^2)}, \ (i,j) \in \{1,...,N\},\ i \neq j \tag{3.57}$$

then the matrix $\partial F(\bar{q}, \bar{l})/\partial \bar{q}\big|_{\bar{q}=\bar{l}}$ is positive definite, which in turn implies that the equilibrium point $\bar{q} = \bar{l}$ is asymptotically stable.

Step 2.2. Proof of $\bar{q} = \bar{q}_c$ being the unstable/saddle equilibrium point(s): The idea is to consider block matrices on the main diagonal of the matrix $\partial F(\bar{q}, \bar{l})/\partial \bar{q}\big|_{\bar{q}=\bar{q}_c}$ and show that there exists at least one block matrix whose determinant is negative. Define $H_{ijc} = \partial \Xi_{ij}/\partial q_{ij}\big|_{\bar{q}=\bar{q}_c}$ and let ϕ_a and ϕ_b be the a^{th} and b^{th} elements of q_{ijc}, $(a,b) \in \{1,...,n\}$, $a \neq b$. We form the matrices H_{ijc}^{ab} from the matrix H_{ijc} as follows

$$H_{ijc}^{ab} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \tag{3.58}$$

where

$$\begin{aligned}
h_{11} &= N + 2\delta k \Pi_{ijc} \beta_{ijc}^{k-1} + 2\delta k[(k-1)\Pi_{ijc}\beta_{ijc}^{k-2} + 2k/\beta_{ijc}^{k+2}]\phi_a^2, \\
h_{12} &= 2\delta k[(k-1)\Pi_{ijc}\beta_{ijc}^{k-2} + 2k/\beta_{ijc}^{k+2}]\phi_a\phi_b, \\
h_{21} &= 2\delta k[(k-1)\Pi_{ijc}\beta_{ijc}^{k-2} + 2k/\beta_{ijc}^{k+2}]\phi_a\phi_b, \\
h_{22} &= N + 2\delta k \Pi_{ijc} \beta_{ijc}^{k-1} + 2\delta k[(k-1)\Pi_{ijc}\beta_{ijc}^{k-2} + 2k/\beta_{ijc}^{k+2}]\phi_b^2
\end{aligned}$$

with $\Pi_{ijc} = 1/\beta_{ijl}^{2k} - 1/\beta_{ijc}^{2k}$. The determinant of H_{ijc}^{ab} is given by

$$\det(H_{ijc}^{ab}) = (N + 2\delta k \Pi_{ijc}\beta_{ijc}^{k-1})\Delta_{ijc}^{ab} \tag{3.59}$$

where

$$\Delta_{ijc}^{ab} = N + 2\delta k \Pi_{ijc}\beta_{ijc}^{k-1} + 2\delta k[(k-1)\Pi_{ijc}\beta_{ijc}^{k-2} + 2k/\beta_{ijc}^{k+2}](\phi_a^2 + \phi_b^2). \tag{3.60}$$

Let us consider the sum:

$$\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} \Delta_{ijc}^{ab} = n(n-1)N + 2\delta k(n-1)(2(k-1)+n)\beta_{ijc}^{k-1}/\beta_{ijl}^{2k} + \\ 2\delta k(n-1)(2(k+1)-n)/\beta_{ijc}^{k+1}. \tag{3.61}$$

Since $n > 1$, picking $k > n/2 - 1$ ensures that $\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} \Delta_{ijc}^{ab} > 0$. Therefore, there exists at least one pair $(a,b) \in \{1,...,n\}$ denoted by (a^*,b^*) such that $\Delta_{ijc}^{a^*b^*} > 0$. Now for all $(i,j) \in \{1,...,N\}$, $i \neq j$ let us consider the sum:

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\det(H_{ijc}^{a^*b^*})}{\Delta_{ijc}^{a^*b^*}}\beta_{ijc} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (N\beta_{ijc} + 2\delta k \Pi_{ijc}\beta_{ijc}^{k}). \tag{3.62}$$

On the other hand, multiplying both sides of $F(\bar{q}_c, \bar{l}) = 0$ with $\bar{q}_c^T$ results in $\bar{q}_c^T F(\bar{q}_c, \bar{l}) = 0$, which is expanded to

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (Nq_{ijc}^T(q_{ijc} - l_{ij}) + 2\delta k N \Pi_{ijc}\beta_{ijc}^{k}) = 0. \tag{3.63}$$

Substituting (3.63) into (3.62) results in

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\det(H_{ijc}^{a^*b^*})}{\Delta_{ijc}^{a^*b^*}}\beta_{ijc} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (N-2)\beta_{ijc} + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} q_{ijc}^T l_{ij}. \tag{3.64}$$

The term $\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \left(q_{ijc}^T l_{ij}\right)$ is strictly negative since at the point where $q_{ij} = l_{ij}$ (the point F in Figure 3.5) all attractive and repulsive forces are equal to zero while at the point where $q_{ij} = q_{ijc}$ (the point C in Figure 3.5) the sum of attractive and repulsive forces is equal to zero. Therefore the point $q_{ij} = 0$ (the point O in Figure 3.5) must locate between the points $q_{ij} = l_{ij}$ and $q_{ij} = q_{ijc}$, see Figure 3.5. Furthermore if we write (3.63) as

$$2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \beta_{ijc} + \delta k(\beta_{ijc}^{k}/\beta_{ijl}^{2k} - 1/\beta_{ijc}^{k}) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} q_{ijc}^T l_{ij} \tag{3.65}$$

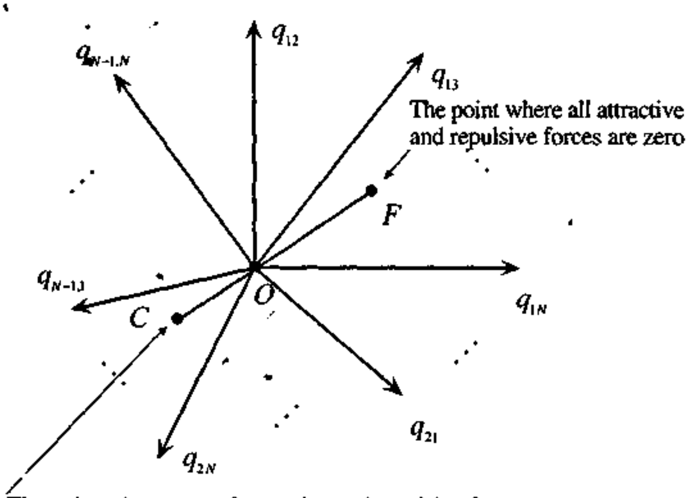


Fig. 3.5. Illustration of location of critical points.

we can see that deceasing δ results in a decrease in β_{ijc} since β_{ijl} is a bounded constant and the right hand side of (3.65) is negative. Therefore, choosing a sufficiently small δ ensures that the right hand of (3.62) is strictly negative since $\beta_{ijc} = 0.5||q_{ijc}||^2$. That is

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\det(H_{ijc}^{a^*b^*})}{\Delta_{ijc}^{a^*b^*}}\beta_{ijc} < 0 \tag{3.66}$$

which implies that there exists at least one pair $(i,j) \in \{1,...,N\}$ denoted by (i^*,j^*) such that

$$\det(H_{i^*j^*c}^{a^*b^*}) < 0 \tag{3.67}$$

which implies that at least one eigenvalue of the matrix $\partial F(\bar{q},\bar{l})/\partial\bar{q}\big|_{\bar{q}=\bar{q}_c}$ is negative. This in turn guarantees that $\bar{q}_c$ is an unstable/saddle equilibrium point of (3.51). Proof of Theorem 3.4 is completed.

3.2.4 Simulations

We carry out a simulation example in two-dimensional space to illustrate the results. The number of robots is $N = 4$. The initial positions of robots are chosen randomly in the circle with a radius of 0.5 centered at the origin. The design constants are chosen as $C = \text{diag}(0.4, 0.4)$, $k = 0.5$, $\delta = 0.1$. It is noted that this choice satisfies the conditions in the proof of Theorem 1. We run two simulations with $u_d = [1\ 0.2]^T$ (linear formation motion meaning that each robot will move on a rectilinear line to form the desired formation) and $u_d = [\sin(0.5t)\ \ \cos(0.5t)]$ (circular formation motion meaning that each robot will move on a circle to form the desired formation). For clarity, we take the formation parameter η as a scalar to implement formation expansion. The desired formation is depicted in Figure 3.6. These simulations are motivated by gradient climbing missions in which the mobile sensor network (each mobile robot serves as a mobile sensor) seeks out local maxima or minima in the environmental field. The network can adapt its configuration in response to the sensed environment in order to optimize its gradient climb. For example, gradients in temperature fields (among others) can be estimated from the data collected by the mobile robots; these are of interest for enabling gradient climbing to locate and track features such as fronts and eddies. These gradients can be used to compute the desired reference velocity vector u_d in our simulations in this section. In the first 4.5 seconds (for the linear formation motion case) and 15 seconds (for the circular formation case), η is set to zero then is updated to $\eta_f = 3$ for the rest of simulation time. The update gain is chosen as $\Gamma = 2$ (scalar).

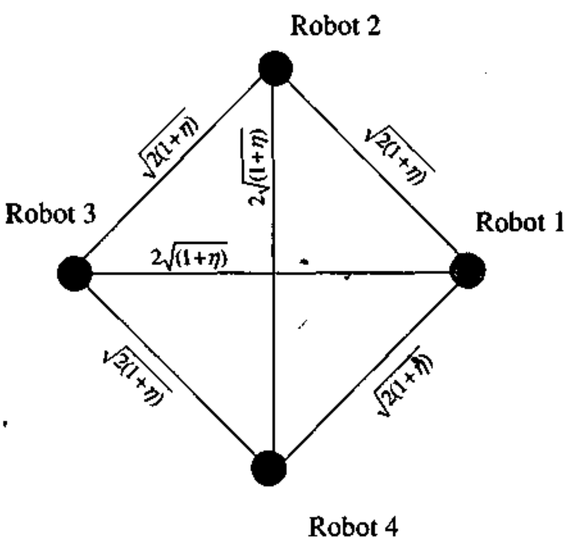


Fig. 3.6. Desired formation for simulation.

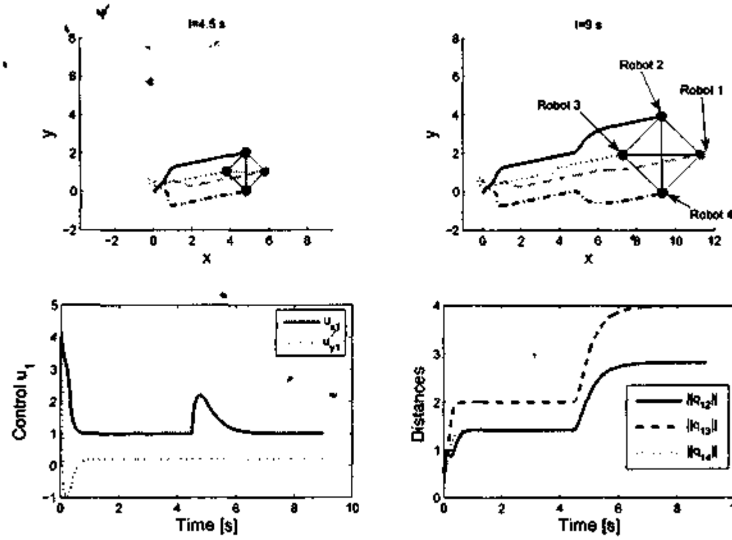


Fig. 3.7. Linear formation motion: simulation results.

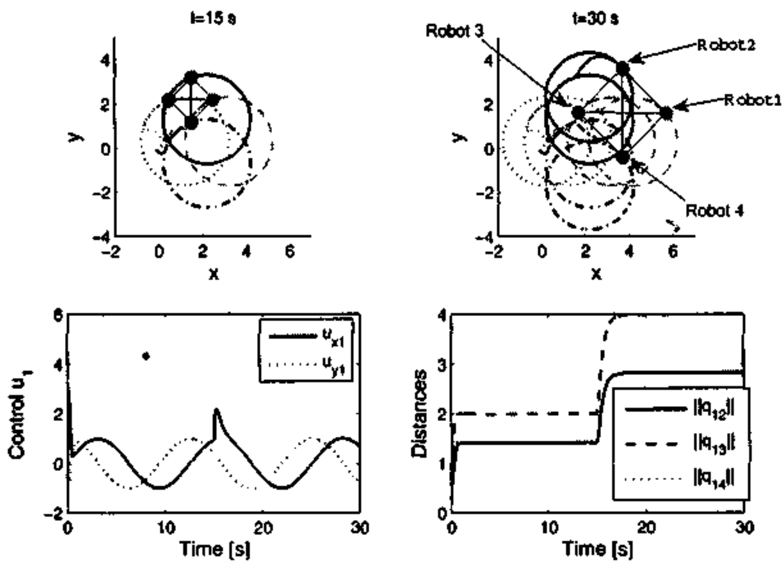


Fig. 3.8. Circular formation motion: simulation results.

Figures 3.7 and 3.8 plot simulation results for the linear formation motion and circular formation cases, respectively. For clarity, we only plot the control $u_1 = [u_{x1}\ u_{y1}]^T$ and distances from the robot 1 to other members in the group, i.e. $\|q_{12}\|$, $\|q_{13}\|$ and $\|q_{14}\|$. It is seen from these figures that the desired formation shapes are nicely achieved and there are no collisions between any robots, see the bottom right figures in Figures 3.7 and 3.8, where the distances from the robot 1 to other members in the groups are plotted. Clearly, these distances are always larger than zero. It is also seen from Figures 3.7 and 3.8 that at the beginning all the robots rapidly move away from each other to avoid collisions since they start pretty close to each other.

3.3 Formation control of N mobile robots

This section shows that the control method developed in the previous sections can be readily extended to force a group of N nonholonomic mobile robots of unicycle type to move in such a way that a desired formation is achieved. For clarity, we consider only the kinematic model of the nonholonomic mobile robots. Designing the control system at the dynamic level even without requiring robot velocities be measured can be carried out using one more backstepping step [12] and the proposed exponential observer in Chapter 2. Consider the kinematic model of the unicycle mobile robot i, whose only two wheels are actuated and the third wheel is not actuated shown in Figure 3.9 (see Chapter 1), given by

$$\begin{aligned}\dot{x}_i &= v_i \cos(\phi_i)\\ \dot{y}_i &= v_i \sin(\phi_i)\\ \dot{\phi}_i &= w_i\end{aligned} \tag{3.68}$$

with $i = 1, ..., N$.

Moreover, we will consider the the linear and angular velocities (v_i and w_i) of the robot i as the control inputs.

3.3.1 Control design

The control design consists of two steps. At the first step, we consider the control v_i and the yaw angle ϕ_i as a virtual control to steer the robot position (x_i, y_i) to its desired location. At the second step, the control w_i will be desired to force the virtual yaw angle to converge to its actual yaw angle.

Step 1. Define

$$\phi_{ei} = \phi_i - \alpha_{\phi_i} \tag{3.69}$$

where α_{ϕ_i} is a virtual control of ϕ_i. With (3.69), we can write (3.68) as

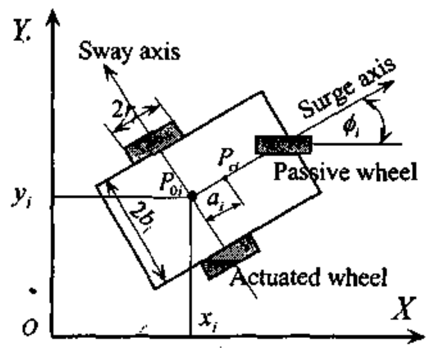


Fig. 3.9. Geometric description of the nonholonomic mobile robot i.

$$\dot{q}_i = u_i + \Lambda_{\phi_{ei}} \tag{3.70}$$

where

$$q_i = \begin{vmatrix} x_i \\ y_i \end{vmatrix}, \quad u_i = \begin{vmatrix} \cos(\alpha_{\phi_i}) \\ \sin(\alpha_{\phi_i}) \end{vmatrix} v_i,$$

$$\Lambda_{\phi_{ei}} = \begin{vmatrix} (\cos(\phi_{ei}) - 1)\cos(\alpha_{\phi_i}) - \sin(\phi_{ei})\sin(\alpha_{\phi_i}) \\ \sin(\phi_{ei})\cos(\alpha_{\phi_i}) + (\cos(\phi_{ei}) - 1)\sin(\alpha_{\phi_i}) \end{vmatrix} v_i. \tag{3.71}$$

It is seen that (3.70) is almost of the same form as (3.21). However, the problem is that the controls v_i and α_{ϕ_i} are not solvable directly from the control u_i if u_i is not designed properly. We therefore present briefly how u_i is designed to tackle that problem. Consider the following potential function (the same form as (3.24))

$$\varphi_i = \gamma_i + \delta\beta_i \tag{3.72}$$

where δ, γ_i and β_i are defined in Section 3.2.2, see (3.26) and (3.27). Differentiating both sides of (3.72) along the solutions of (3.70) gives

$$\dot{\varphi}_i = \sum_{j \in N_i} [\Omega_{ij}^T(u_i + \Lambda_{\phi_{ei}} - u_d^* - (u_j + \Lambda_{\phi_{ej}} - u_d^*)) - \Psi_{ij}^T \dot{\eta}] \tag{3.73}$$

where Ω_{ij} and Ψ_{ij} are defined in (3.29), and $u_d^* = \sqrt{1 + \sum_{i=1}^{N} || \sum_{j \in N_i} \Omega_{ij}||^2}\, u_d$. It is noted that we use u_d^* instead of u_d in (3.73) to overcome the nonholonomic problem of the mobile robot under investigation. Indeed, $\lim_{t \to \infty} \sum_{j \in N_i} \Omega_{ij}(t) = 0$ implies that $\lim_{t \to \infty} u_d^*(t) = u_d$. From (3.73), we choose the control u_i and the update law for η as

$$\begin{aligned} u_i &= -C\|u_d\| \sum_{j \in N_i} \Omega_{ij} + u_d^* \\ \dot{\eta} &= -\Gamma(\eta - \eta_f) \end{aligned} \tag{3.74}$$

where C and Γ are diagonal positive definite matrices. Again, $\|u_d\|$ is included in the control u_i to overcome the nonholonomic problem. Defining $\phi_d = \arctan(u_{dy}/u_{dx})$, then from the first equations of (3.74) and (3.71), we have

$$\begin{aligned} \cos(\alpha_{\phi_i})v_i &= -c_1\|u_d\| \sum_{j \in N_i} \Omega_{xij} + \sqrt{1 + \sum_{i=1}^{N} \| \sum_{j \in N_i} \Omega_{ij}\|^2}\|u_d\| \cos(\phi_d) \\ \sin(\alpha_{\phi_i})v_i &= -c_2\|u_d\| \sum_{j \in N_i} \Omega_{yij} + \sqrt{1 + \sum_{i=1}^{N} \| \sum_{j \in N_i} \Omega_{ij}\|^2}\|u_d\| \sin(\phi_d) \end{aligned} \tag{3.75}$$

where Ω_{xij} and Ω_{yij} are defined as $\Omega_{ij} = [\,\Omega_{xij} \;\; \Omega_{yij}]^T$, c_1 and c_2 are defined as $C = \text{diag}(c_1, c_2)$. We now need to solve (3.75) for v_i and α_{ϕ_i}. To do this, multiplying both sides of the first and second equations of (3.75) with $\cos(\phi_d)$ and $\sin(\phi_d)$, respectively, then adding them together result in

$$\begin{aligned} \cos(\alpha_{\phi_i} - \phi_d)v_i = &-c_1\|u_d\| \sum_{j \in N_i} \Omega_{xij}\cos(\phi_d) - \\ & c_2\|u_d\| \sum_{j \in N_i} \Omega_{yij}\sin(\phi_d) + \sqrt{1 + \sum_{i=1}^{N} \| \sum_{j \in N_i} \Omega_{ij}\|^2}\|u_d\|. \end{aligned} \tag{3.76}$$

On the other hand, multiplying both sides of the first and second equations of (3.75) with $\sin(\phi_d)$ and $\cos(\phi_d)$, respectively, then subtracting from each other result in

$$\sin(\alpha_{\phi_i} - \phi_d)v_i = c_1\|u_d\| \sum_{j \in N_i} \Omega_{xij}\sin(\phi_d) - c_2\|u_d\| \sum_{j \in N_i} \Omega_{yij}\cos(\phi_d). \tag{3.77}$$

From (3.76) and (3.77), we have

$$\alpha_{\phi_i} = \phi_d + \arctan\left(\frac{c_1 \sum_{j \in N_i} \Omega_{xij}\sin(\phi_d) - c_2 \sum_{j \in N_i} \Omega_{yij}\cos(\phi_d)}{- \sum_{j \in N_i} (c_1\Omega_{xij}\cos(\phi_d) + c_2\Omega_{yij}\sin(\phi_d)) + \sqrt{1 + \sum_{i=1}^{N} \| \sum_{j \in N_i} \Omega_{ij}\|^2}} \right). \tag{3.78}$$

It is seen that (3.78) is well-defined if the positive constants c_1 and c_2 are chosen such that

$$c_1 + c_2 < 1. \tag{3.79}$$

The control v_i is found by solving (3.75) as

$$\begin{aligned} v_i = {} & \cos(\alpha_{\phi_i})\|u_d\| \left(-c_1 \sum_{j \in N_i} \Omega_{xij} + \sqrt{1 + \sum_{i=1}^{N} \|\sum_{j \in N_i} \Omega_{ij}\|^2} \cos(\phi_d)\right) + \\ & \sin(\alpha_{\phi_i})\|u_d\| \left(-c_2 \sum_{j \in N_i} \Omega_{yij} + \sqrt{1 + \sum_{i=1}^{N} \|\sum_{j \in N_i} \Omega_{ij}\|^2} \sin(\phi_d)\right). \end{aligned} \tag{3.80}$$

Substituting (3.74) into (3.73) results in

$$\dot{\varphi}_i = -\|u_d\| \sum_{j \in N_i} \Omega_{ij}^T C \sum_{j \in N_i} \Omega_{ij} + \sum_{j \in N_i} [\Omega_{ij}^T (\Lambda_{\phi_{ei}} - (u_j + \Lambda_{\phi_{ej}} - u_d^*)) + \Psi_{ij}^T \Gamma(\eta - \eta_f)]. \tag{3.81}$$

Step 2. To design the control w_i, differentiating both sides of (3.69) along the solutions of the third equation of (3.68) and choosing the control w_i as

$$w_i = -d_i \phi_{ei} - \dot{\alpha}_{\phi_i} - \sum_{j \in N_i} \Omega_{ij}^T \Lambda_{\phi_{ei}} / \phi_{ei} \tag{3.82}$$

where d_i is a positive constant, and the term $\sum_{j \in N_i} \Omega_{ij}^T \Lambda_{\phi_{ei}} / \phi_{ei}$ is to cancel the cross term $\sum_{j \in N_i} \Omega_{ij}^T \Lambda_{\phi_{ei}}$ in (3.81), result in

$$\dot{\phi}_{ei} = -d_i \phi_{ei} - \sum_{j \in N_i} \Omega_{ij}^T \Lambda_{\phi_{ei}} / \phi_{ei}. \tag{3.83}$$

Note that $\Lambda_{\phi_{ei}} / \phi_{ei}$ is well defined since $\sin(\phi_{ie})/\phi_{ie} = \int_0^1 \cos(\phi_{ie}\lambda) d\lambda$ and $(\cos(\phi_{ie}) - 1)/\phi_{ie} = \int_0^1 \sin(\phi_{ie}\lambda) d\lambda$ are smooth functions.

3.3.2 Stability analysis

We consider the following function

$$\varphi_{tot} = \log(1 + \sum_{i=1}^{N} (\varphi_i + \phi_{ei}^2)) + \frac{1}{2}(\eta - \eta_f)^T \Gamma(\eta - \eta_f) \tag{3.84}$$

whose derivative along the solutions of (3.81), (3.83) and the second equation of (3.74) satisfies

$$\dot{\varphi}_{tot} = -2\frac{\|u_d\|\sum_{i=1}^{N}\sum_{j\in N_i}\Omega_{ij}^T C\sum_{j\in N_i}\Omega_{ij} + \sum_{i=1}^{N} d_i\phi_{ei}^2}{1+\sum_{i=1}^{N}(\varphi_i+\phi_{ei}^2)} + \frac{\sum_{i=1}^{N}\sum_{j\in N_i}\Psi_{ij}^T\Gamma(\eta-\eta_f)}{1+\sum_{i=1}^{N}(\varphi_i+\phi_{ei}^2)} - (\eta-\eta_f)^T\Gamma(\eta-\eta_f) \tag{3.85}$$

where we have used

$$\begin{aligned} -\sum_{i=1}^{N}\sum_{j\in N_i}[\Omega_{ij}^T(u_j+\Lambda_{\phi_{ej}}-u_d^*) &= \sum_{i=1}^{N}\sum_{j\in N_i}[\Omega_{ij}^T(u_i-u_d^*+\Lambda_{\phi_{ei}}) \\ &= -\|u_d\|\sum_{i=1}^{N}\sum_{j\in N_i}\Omega_{ij}^T C\sum_{j\in N_i}\Omega_{ij}+\sum_{i=1}^{N}\sum_{j\in N_i}\Omega_{ij}^T\Lambda_{\phi_{ei}}. \end{aligned} \tag{3.86}$$

The rest of stability analysis can be carried out in the same lines as in Proof of Theorem 3.4 since (3.85) is of the same form as (3.38) and $\lim_{t\to\infty}\|u_d(t)\| \neq 0$ by assumption. Finally, note that $\lim_{t\to\infty}\phi_{ei}(t) = 0$ and $\lim_{t\to\infty}\sum_{j\in N_i}\Omega_{ij}(t) = 0$ implies that $\lim_{t\to\infty}(\phi_i(t)-\phi_d) = 0$, i.e. the yaw angle of all robots converge to the desired angle $\phi_d = \arctan(u_{dy}/u_{dx})$.

3.3.3 Simulation results

We now perform a simulation to illustrate the results in the previous subsection. The number of robots, initial conditions of the robot positions, control gains, desired formation velocity and desired formation shape are the same as in Section 3.2.4. The robot heading angles are initialized randomly in the circle with a radius of 0.5 centered at the origin. For clarity, we only simulate the circular formation motion, and we do not include simulation results on the formation expansion as in Section 3.2.4, i.e. the formation parameter η is set to zero in all the simulation time. The other design constants are chosen as $d_i = 5$. Simulation results are plotted in Figure 3.10. Again, it is seen that the robots are forced to move to nicely achieve the desired formation and no collisions between the robots occur. Moreover, the yaw angle of all robots converges to the desired value ϕ_d, see the top-right figure in Figure 3.10, where the yaw angle errors are plotted ($\phi_{ei} = \phi_i - \phi_d$). A close look at Figure 3.10 shows that the main different between simulation results in this subsection and those in Section 3.2.4 is that the robots take a longer time to approach the desired formation. This is because we use the heading angles ϕ_i as the virtual controls to steer the robots to overcome the noholonomic constraint.

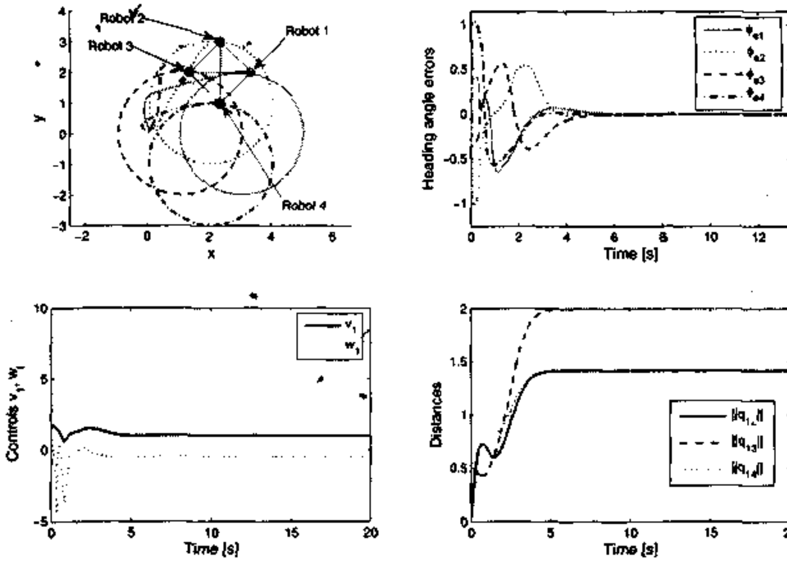


Fig. 3.10. Mobile robot circular formation motion: simulation result.

3.4 Notes and references

Basically, formation control involves the control of positions of a group of the agents such that they stabilize/track desired locations relative to reference point(s), which can be another agent(s) within the team, and can either be stationary or moving. Three popular approaches to formation control are leader-following (e.g. [39], [36]), behavioral (e.g. [40], [41]), and use of virtual structures (e.g. [42], [43]). Most research works investigating formation control utilize one or more of these approaches in either a centralized or decentralized manner. Centralized control schemes, see e.g. [36] and [44], use a single controller that generates collision free trajectories in the workspace. Although these guarantee a complete solution, centralized schemes require high computational power and are not robust due to the heavy dependence on a single controller. On the other hand, decentralized schemes, see e.g. [45], [46] and [37], require less computational effort, and are relatively more scalable to the team size. The decentralized approach usually involves a combination of agent based local potential fields ([36], [37], [38]. The main problem with the decentralized approach, when collision avoidance is taken into account, is that it is extremely difficult to predict and control the critical points (the controlled system often has multiple equilibrium points). It is difficult to design a controller such that all the equilibrium points except for the desired equilibrium ones are unstable points. Recently, a method based on a different navigation function from [31] provided a centralized formation stabilization control design strategy is proposed in [46]. This work is extended to a decentralized version in [47]. However, the navigation function approaches a finite value when a collision occurs. In [32], [31], [46] and [47], the tuning constants, which are crucial to guarantee that the only desired equilibrium points are asymptotic stable and that the other critical points are unstable, cannot be obtained explicitly but "are chosen sufficiently small". When it comes to a practical implementation, an important issue is "how small these constants should be?" Moreover, the control design methods ([36], [48], [37]) based on the potential/navigation functions that are equal to infinity when a collision occurs exhibit very large control efforts if the agents are close to each other. Hence, a bounded control is called for. These problems motivated the work in this chapter. The material in this chapter is based on the work in [49] and [50].

4

Formation Control of Mobile Robots with Unlimited Sensing: Output Feedback

In this chapter, we investigate formation control of a group of unicycle-type mobile robots with a little amount of inter-robot communication. A combination of the virtual structure and path-tracking approaches is used to derive the formation architecture. Each individual robot has only position and orientation available for feedback. For each robot, a coordinate transformation is first derived to cancel the velocity quadratic terms. An observer is then designed to globally exponentially/asymptotically estimate the unmeasured velocities. An output feedback controller is designed for each robot in such a way that the derivative of the path parameter is left as a free input to synchronize the robots' motion. Simulations illustrate the soundness of the proposed controller.

4.1 Problem statement

4.1.1 Formation setup

A group of N mobile robots needs N individual reference paths, which are parameterized so that when all the path parameters are synchronized, the robots are in formation. A modification of the conventional virtual structure approach is used here to generate the reference paths. We consider a virtual structure whose center moves along a reference path $\Gamma_0(s_0) = [x_{d0}(s_0),\ y_{d0}(s_0)]^T$ with s_0 being tne path parameter, see Figure 4.1. Since the structure under consideration is virtual, the center does not have to be the center of gravity but can be any convenient point. The shape of the virtual structure can be varied by specifying the distance $l_i(x_{d0}(s_i), y_{d0}(s_i))$ from each place-holder to the center of the structure. This variance is particular meaningful in practice. For example, the formation has to change its shape when the vehicles enter a tunnel. When the structure moves along the path $\Gamma_0(s_0)$, the place-holders generate

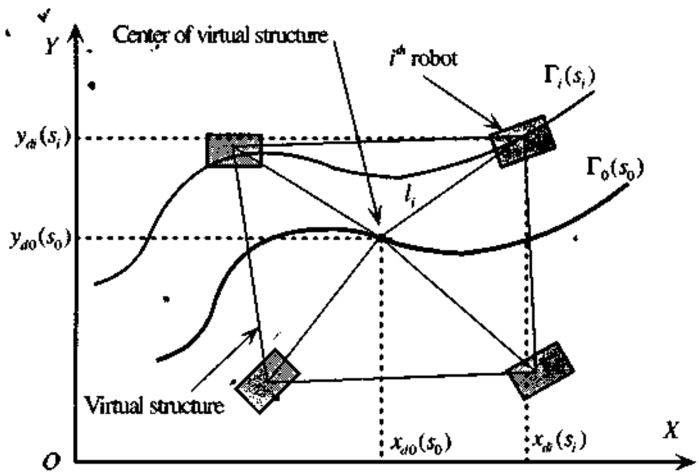


Fig. 4.1. Formation coordinates.

reference paths $\Gamma_i(s_i) = [x_{di}(s_i),\ y_{di}(s_i)]^T$, $1 \le i \le N$ with s_i being the i^{th} path parameter, given by:

$$\Gamma_i(s_i) = \Gamma_0(s_i) + R(\phi_{d0}(s_i))l_i(x_{d0}(s_i), y_{d0}(s_i)) \tag{4.1}$$

where

$$\begin{aligned}
&l_i(x_{d0}(s_i), y_{d0}(s_i)) = \begin{bmatrix} l_{xi}(x_{d0}(s_i), y_{d0}(s_i)) \\ l_{yi}(x_{d0}(s_i), y_{d0}(s_i)) \end{bmatrix}, \\
&R(\phi_{d0}(s_i)) = \begin{bmatrix} \cos(\phi_{d0}(s_i)) & -\sin(\phi_{d0}(s_i)) \\ \sin(\phi_{d0}(s_i)) & \cos(\phi_{d0}(s_i)) \end{bmatrix}, \\
&\phi_{d0}(s_i) = \arctan\left(\frac{y'_{d0}(s_i)}{x'_{d0}(s_i)}\right), \\
&x'_{d0}(s_i) = \frac{\partial x_{d0}(s_0)}{\partial s_0}\bigg|_{s_0=s_i}, \quad y'_{d0}(s_i) = \frac{\partial y_{d0}(s_0)}{\partial s_0}\bigg|_{s_0=s_i}.
\end{aligned} \tag{4.2}$$

Intuitively, one might set $s_i(t) = t$, i.e. the current time instant t is used as the path parameter. However, the use of s_i as the path parameter has an advantage that its time evolution, $\dot{s}_i$, can be treated as an additional "control input". Tnis additional control input is utilized so that the overall control system for single mobile robots possesses certain robustness with respect to measurement error and external disturbances. In this chapter, we will use this additional control input for formation feedback and synchronization of the path parameters. It is noted that in the conventional virtual structure approach, the distance from each place-holder to the structure center is constant, i.e. the shape of the structure cannot be changed. Having generated each reference path for each mobile robot, the remaining tasks are to design a controller such that each vehicle tracks its own desired path, and all the path parameters of the reference paths are synchronized.

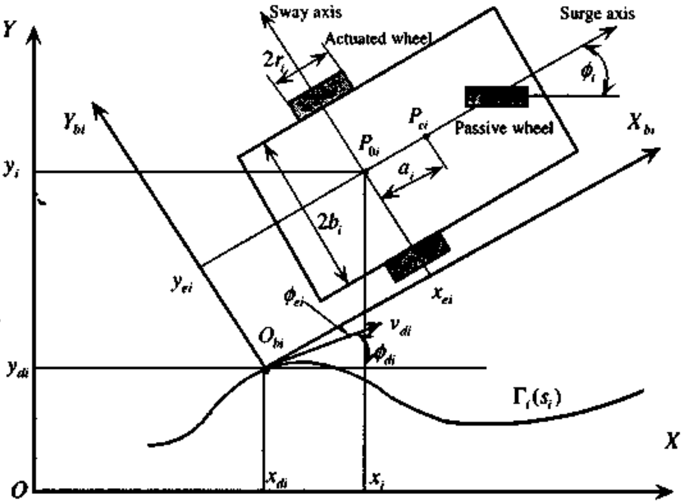


Fig. 4.2. The i^{th} mobile robot and interpretation of path tracking errors.

4.1.2 Mobile robot dynamics

We consider a mobile robot with two actuated wheels in Chapter 1, see Figure 4.2. For convenience, we rewrite the equations of motion of the i^{th} mobile robot here:

$$\begin{aligned} &\dot{\eta}_i = J_i(\eta_i)\omega_i \\ &M_i\dot{\omega}_i + C_i(\dot{\eta}_i)\omega_i + D_i\omega_i = \tau_i, \quad i = 1, ..., N \end{aligned} \tag{4.3}$$

where all the state variables and parameters are defined in Section 1.3.1, Chapter 1.

4.1.3 Control objective

Before presenting the control objective, we impose the following assumption:

Assumption 4.1 *1) For each robot, only position $(x_i, y_i)$and orientation ϕ_i are measurable.*
2) For each value of s_i, there exists a unique value of $x_{di}(s_i)$ and $y_{di}(s_i)$.
3) There exist strictly positive constants ε_{1i} and ε_{2i} such that:

$$x'^2_{di} + y'^2_{di} \geq \varepsilon_{1i}, \quad \dot{s}_0 \geq \varepsilon_{2i} \tag{4.4}$$

where $\bullet'$ denoting $\partial \bullet / \partial s_i$.

Remark 4.2. Part 1) of Assumption 4.1 implies that we need to design an output feedback control system. Part 2) means the unique solvability of each individual path from its parameter. Part 3) means that each path is regular, and that the virtual structure moves forward. The case where the virtual structure moves backward can be treated in similar way. Part 3) also means that we do not consider the problem of stabilization.

Having discussed on how the reference path, see (4.1), for each individual mobile robot, the formation control objective boils down to design a control system, under Assumption 4.1, such that

$$\lim_{t\to\infty} \|((x_i(t) - x_{di}(t)), (y_i(t) - y_{di}(t)), (\phi_i(t) - \phi_{di}(t)))\| = 0 \tag{4.5}$$

and

$$\lim_{t\to\infty} (s_i(t) - s_0(t)) = 0 \tag{4.6}$$

where $\phi_{di} = \arctan\left(\frac{y'_{di}(s_i)}{x'_{di}(s_i)}\right)$. The objective (4.5) justifies both position and orientation tracking tasks in the sense that each robot moves on the path and its linear velocity is tangential to its own path. On the other hand, the objective (4.6) guarantees synchronization of all the path parameters, i.e. all the robots are properly arranged.

4.2 Observer design

If all robots in the group are damped, i.e. the damping matrix D_i is strictly positive definite, then one can use the observer designed in Chapter 2, Section 2.2.2. For a more general result, we do not assume any robots be damped or undamped in advance. For the sake of completeness, we represent some part of the observer design in Chapter 2, Section 2.2.2 including the coordinate transformation to cancel the velocity cross terms.

We now convert the wheel velocities ω_{1i} and ω_{2i} to the linear, v_i, and angular, w_i, velocities of the robot by the relationship:

$$\vartheta_i = B_i^{-1}\omega_i, \text{ with } \vartheta_i = [v_i \; w_i]^T, \; B_i = \frac{1}{r_i}\begin{bmatrix} 1 & b_i \\ 1 & -b_i \end{bmatrix}. \tag{4.7}$$

With (4.7), we write (4.3) as

$$\begin{aligned} \dot{\eta}_i &= \bar{J}_i(\eta_i)\vartheta_i \\ \dot{\vartheta}_i &= -\bar{C}_i(w_i)\vartheta_i - \bar{D}_i\vartheta_i + B_i^{-1}M_i^{-1}\tau_i \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \bar{J}_i(\eta_i) &= \begin{bmatrix} \cos(\phi_i) & \sin(\phi_i) & 0 \\ 0 & 0 & 1 \end{bmatrix}^T, \\ \bar{C}_i &= \begin{bmatrix} 0 & -b_ic_i/(m_{11i}+m_{12i})w_i \\ c_i/b_i/(m_{11i}-m_{12i})w_i & 0 \end{bmatrix}, \\ \bar{D}_i &= B_i^{-1}M_i^{-1}D_iB_i. \end{aligned} \tag{4.9}$$

We first remove the quadratic velocity terms in the second equation of (4.8) by introducing the following coordinate change:

$$X_i = Q_i(\eta_i)\vartheta_i \tag{4.10}$$

where $Q_i(\eta_i)$ is a globally invertible matrix with bounded elements to be determined. Using (4.10), we write the second equation of (4.8) as follows:

$$\dot{X}_i = [\dot{Q}_i(\eta_i)\vartheta_i - Q_i(\eta_i)\bar{C}_i(w_i)\vartheta_i] + Q_i(\eta_i)(-\bar{D}_i\vartheta_i + B_i^{-1}M_i^{-1}\tau_i) \tag{4.11}$$

We now cancel the square bracket in the right hand side of (4.11) for all $(\eta_i, \vartheta_i) \in \mathbb{R}^5$. We assume that $q_{ikj}(\eta_i)$, $k = 1,2$, $j = 1,2$ are the elements of $Q_i(\eta_i)$. We now need to find $q_{ikj}(\eta_i)$, $k = 1,2$, $j = 1,2$ such that

$$[\dot{Q}_i(\eta_i)\vartheta_i - Q_i(\eta_i)\bar{C}_i(w_i)\vartheta_i] = 0, \quad \forall(\eta_i, \vartheta_i) \in \mathbb{R}^5. \tag{4.12}$$

Using the first equation of (4.8), we can write (4.12) as

$$\begin{bmatrix} \frac{\partial q_{i11}}{\partial x_i}\cos(\phi_i)v_i + \frac{\partial q_{i11}}{\partial y_i}\sin(\phi_i)v_i + \frac{\partial q_{i11}}{\partial \phi_i}w_i & \frac{\partial q_{i12}}{\partial x_i}\cos(\phi_i)v_i + \frac{\partial q_{i12}}{\partial y_i}\sin(\phi_i)v_i + \frac{\partial q_{i12}}{\partial \phi_i}w_i \\ \frac{\partial q_{i21}}{\partial x_i}\cos(\phi_i)v_i + \frac{\partial q_{i21}}{\partial y_i}\sin(\phi_i)v_i + \frac{\partial q_{i21}}{\partial \phi_i}w_i & \frac{\partial q_{i22}}{\partial x_i}\cos(\phi_i)v_i + \frac{\partial q_{i22}}{\partial y_i}\sin(\phi_i)v_i + \frac{\partial q_{i22}}{\partial \phi_i}w_i \end{bmatrix} \times \begin{bmatrix} v_i \\ w_i \end{bmatrix} - \begin{bmatrix} q_{i11} & q_{i12} \\ q_{i21} & q_{i22} \end{bmatrix} \begin{bmatrix} -\frac{b_ic_i}{m_{11i}+m_{12i}}w_i^2 \\ \frac{c_i}{b_i(m_{11i}-m_{12i})}v_iw_i \end{bmatrix} = 0 \tag{4.13}$$

which further yields

$$\begin{aligned} &\left(\frac{\partial q_{ik1}}{\partial x_i}\cos(\phi_i) + \frac{\partial q_{ik1}}{\partial y_i}\sin(\phi_i)\right)v_i^2 + \\ &\left(\frac{\partial q_{ik1}}{\delta \phi_i} + \frac{\partial q_{ik2}}{\partial x_i}\cos(\phi_i) + \frac{\partial q_{ik2}}{\partial y_i}\sin(\phi_i) - \frac{c_iq_{ik2}}{b_i(m_{11i}-m_{12i})}\right)v_iw_i + \\ &\left(\frac{\partial q_{ik2}}{\partial \phi_i} + \frac{b_ic_iq_{ik1}}{m_{11i}+m_{12i}}\right)w_i^2 = 0 \end{aligned} \tag{4.14}$$

for $k = 1, 2$. Clearly (4.14) holds if we choose $q_{ikj}(\eta_i)$, $k = 1, 2$, $j = 1, 2$ such that

$$\begin{aligned} &\frac{\partial q_{ik1}}{\partial x_i}\cos(\phi_i) + \frac{\partial q_{ik1}}{\partial y_i}\sin(\phi_i) = 0, \\ &\frac{\partial q_{ik1}}{\partial \phi_i} + \frac{\partial q_{ik2}}{\partial x_i}\cos(\phi_i) + \frac{\partial q_{ik2}}{\partial y_i}\sin(\phi_i) - \frac{c_i q_{ik2}}{b_i(m_{11i} - m_{12i})} = 0, \\ &\frac{\partial q_{ik2}}{\partial \phi_i} + \frac{b_i c_i q_{ik1}}{m_{11i} + m_{12i}} = 0. \end{aligned} \tag{4.15}$$

A family of solutions of the above partial differential equations (PDEs) is

$$\begin{aligned} q_{ik1} &= C_{ik1}\sin(c_i \Delta_i \phi_i) + C_{ik2}\cos(c_i \Delta_i \phi_i), \\ q_{ik2} &= b_i(m_{11i} - m_{12i})\Delta_i \left(C_{ik1}\cos(c_i \Delta_i \phi_i) - C_{ik2}\sin(c_i \Delta_i \phi_i)\right) \end{aligned} \tag{4.16}$$

where $\Delta_i = 1/\sqrt{m_{11i}^2 - m_{12i}^2}$, C_{ik1} and C_{ik2} are arbitrary constants. Setting $C_{i11} = C_{i22} = 0, C_{i12} = C_{i21} = 1$ results in

$$Q_i(\eta_i) = \begin{bmatrix} q_{i11} & q_{i12} \\ q_{i21} & q_{i22} \end{bmatrix} = \begin{bmatrix} \cos(c_i \Delta_i \phi_i) & -b_i \Delta_i (m_{11i} - m_{12i})\sin(c_i \Delta_i \phi_i) \\ \sin(c_i \Delta_i \phi_i) & b_i \Delta_i (m_{11i} - m_{12i})\cos(c_i \Delta_i \phi_i) \end{bmatrix}. \tag{4.17}$$

This matrix is globally invertible and its elements are bounded. We write (4.8) in the (η_i, X_i) coordinates as

$$\begin{aligned} \dot{\eta}_i &= G_i(\eta_i) X_i \\ \dot{X}_i &= -D_{\eta i}(\eta_i) X_i + Q_i(\eta_i) B_i^{-1} M_i^{-1} \tau_i \end{aligned} \tag{4.18}$$

where $G_i(\eta_i) = \bar{J}_i(\eta_i) Q_i^{-1}(\eta_i)$, $D_{\eta i}(\eta_i) = Q_i(\eta_i) \bar{D}_i Q_i^{-1}(\eta_i)$. It is seen that (4.18) is linear in the unmeasured states. If the robot is internally damped a reduced-order observer, i.e. an observer estimates X_i only, can be designed. To cover both damped and un-damped cases, we use the following full-order observer:

$$\begin{aligned} \dot{\hat{\eta}}_i &= G_i(\eta_i)\hat{X}_i + K_{01i}(\eta_i - \hat{\eta}_i) \\ \dot{\hat{X}}_i &= -D_{\eta i}(\eta_i)\hat{X}_i + Q_i(\eta_i) B_i^{-1} M_i^{-1} \tau_i + K_{02i}(\eta_i - \hat{\eta}_i) \end{aligned} \tag{4.19}$$

where $\hat{\eta}_i$ and $\hat{X}_i$ are the estimates of η_i and X_i, respectively. The observer gain matrices K_{01i} and K_{02i} are chosen such that $Q_{01i} = K_{01i}^T P_{01i} + P_{01i} K_{01i}$ is positive definite, and that

$$G_i(\eta_i)^T P_{01i} - P_{02i} K_{02i} = 0 \tag{4.20}$$

with P_{01i} and P_{02i} being diagonal positive definite matrices. From (4.19) and (4.18), we have

$$\begin{aligned} \dot{\tilde{\eta}}_i &= G_i(\eta_i)\tilde{X}_i - K_{01i}\tilde{\eta}_i, \\ \dot{\tilde{X}}_i &= -D_{\eta i}(\eta_i)\tilde{X}_i - K_{02i}\tilde{\eta}_i \end{aligned} \tag{4.21}$$

where $\tilde{\eta}_i = \eta_i - \hat{\eta}_i$, $\tilde{X}_i = X_i - \hat{X}_i$. The aforementioned observer result is summarized in the following theorem.

Theorem 4.3. *Assume that the solutions of (4.18) exist, if the mobile robot is internally damped, i.e. the damping coefficients, d_{11i} and d_{22i}, are strictly positive constants, the observer error dynamics (4.21) is globally exponentially stable at the origin for all $\eta_i \in \mathbb{R}^3$. If the mobile robot is internally un-damped, i.e. $d_{11i} = d_{22i} = 0$, the observer error dynamics (4.21) is globally stable at the origin for all $\eta_i \in \mathbb{R}^3$. Furthermore, assume that the velocity $w_i(t)$ starting at $w_i(t_0)$, $0 \leq t_0 \leq t$ is globally bounded, i.e. there exist a class-K function β_{0i} and a constant c_{0i} such that $|w_i(t)| \leq \beta_{0i}(\|(\eta_i(t_0), v_i(t_0))\|) + c_{0i} := w_{Mi}, \forall\, 0 \leq t_0 \leq t, \quad (\eta_i(t_0), v_i(t_0)) \in \mathbb{R}^5$. Then the origin of (4.21) is globally asymptotically stable for all $\eta_i \in \mathbb{R}^3$.*

Remark 4.4. The assumptions on existence of the solutions of (4.18) and on boundedness of $w_i(t)$ are needed here to establish global exponential/asymptotic stability of (4.21). These assumptions will be relaxed when considering the overall closed loop system, i.e. when an output-feedback controller is introduced.

By defining $\hat{\vartheta}_i = [\hat{v}_i \;\; \hat{w}_i]^T$ being an estimate of the velocity vector ϑ_i as

$$\hat{\vartheta}_i = Q_i^{-1}(\eta_i)\hat{X}_i \tag{4.22}$$

the velocity estimate error vector, $\tilde{\vartheta}_i = [\tilde{v}_i \;\; \tilde{w}_i]^T = \vartheta_i - \hat{\vartheta}_i$ satisfies

$$\tilde{\vartheta}_i = Q_i^{-1}(\eta_i)\tilde{X}_i. \tag{4.23}$$

By differentiating both sides of (4.22) along the solutions of the second equation of (4.19), and noting from (4.22) that $\hat{X}_i = Q_i(\eta_i)\hat{\vartheta}_i$, we have

$$\begin{aligned}
\dot{\hat{\vartheta}}_i = & \frac{dQ_i^{-1}(\eta_i)}{dt}\hat{X}_i - Q_i^{-1}(\eta_i)\left(D_{\eta i}(\eta_i)\hat{X}_i - Q_i(\eta_i)B_i^{-1}M_i^{-1}\tau_i - K_{02i}(\eta_i - \hat{\eta}_i)\right) \\
= & -\bar{C}_i(w_i)\hat{\vartheta}_i - \bar{D}_i\hat{v}_i + B_i^{-1}M_i^{-1}\tau_i + Q_i^{-1}(\eta_i)K_{02i}(\eta_i - \hat{\eta}_i) \\
= & -\bar{C}_i(\hat{w}_i)\hat{\vartheta}_i - \bar{D}_i\hat{\vartheta}_i + B_i^{-1}M_i^{-1}\tau_i - \bar{C}_i(\tilde{w}_i)\hat{\vartheta}_i + Q_i^{-1}(\eta_i)K_{02i}(\eta_i - \hat{\eta}_i)
\end{aligned} \tag{4.24}$$

since $\bar{C}_i(w_i)$ is linear in w_i, see (4.9). We now write (4.8) in conjunction with (4.24) as

$$\begin{aligned}
\dot{x}_i &= \cos(\phi_i)\hat{v}_i + \cos(\phi_i)\tilde{v}_i \\
\dot{y}_i &= \sin(\phi_i)\hat{v}_i + \sin(\phi_i)\tilde{v}_i \\
\dot{\phi}_i &= \hat{w}_i + \tilde{w}_i \\
\dot{\hat{v}}_i &= \tau_{vci} + \Omega_{vi} \\
\dot{\hat{w}}_i &= \tau_{wci} + \Omega_{wi}
\end{aligned} \tag{4.25}$$

where Ω_{vi} and Ω_{wi} are the first and second rows of Ω_i:

$$\Omega_i = -\bar{C}_i(\tilde{w}_i)\hat{\vartheta}_i + Q_i^{-1}(\eta_i)K_{02i}\tilde{\eta}_i \tag{4.26}$$

and we have chosen the control torque

$$\tau_i = M_i B_i \left(\bar{C}_i(\hat{w}_i)\hat{\vartheta}_i + \bar{D}_i\hat{\vartheta}_i + [\tau_{vci} \ \ \tau_{wci}]^T \right) \tag{4.27}$$

with τ_{vci} and τ_{wci} being the new control inputs to be designed in the next section.

4.3 Proof of Theorem 4.3

We prove Theorem 4.3 by considering the damped and un-damped cases separately.

4.3.1 Damped case.

Consider the Lyapunov function

$$V_{01i} = \tilde{\eta}_i^T P_{01i} \tilde{\eta}_i + \tilde{X}_i^T P_{02i} \tilde{X}_i \tag{4.28}$$

whose derivative along the solution of (4.21) and using (4.20) satisfies

$$\dot{V}_{01i} = -\tilde{\eta}_i^T Q_{01i} \tilde{\eta}_i - \tilde{X}_i^T Q_{02i} \tilde{X}_i \tag{4.29}$$

where $Q_{02i} = D_{\eta i}^T(\eta_i) P_{02i} + P_{02i} D_{\eta i}(\eta_i)$. Since D_i is positive definite and $D_{\eta i}(\eta_i) = Q_i(\eta_i) B_i^{-1} M_i^{-1} D_i B_i Q_i^{-1}(\eta_i)$ with $Q_i(\eta_i)$ being given in (4.17), and Q_{01i} is a constant positive definite matrix, there exist strictly positive constants σ_{0i} and ρ_{0i} such that $\left\|(\tilde{\eta}_i(t), \tilde{X}_i(t))\right\| \leq \rho_{0i} \left\|(\tilde{\eta}_i(t_0), \tilde{X}_i(t_0))\right\| e^{-\sigma_{0i}(t-t_0)}$ for all $t \geq t_0 \geq 0$. Global exponential stability of (4.21) holds for all $\eta_i \in \mathbb{R}^3$ because all elements of $\bar{J}_i(\eta_i), Q_i^{-1}(\eta_i), D_{\eta i}(\eta_i)$ are bounded, independent from x_i, y_i, and depend on only $\sin(\phi_i), \cos(\phi_i), \sin(c_i\Delta_i\phi_i), \cos(c_i\Delta_i\phi_i)$ and the robot parameters, as long as the solutions of (4.18) exist. This also implies from (4.23) that

$$\|(\tilde{v}_i(t), \tilde{w}_i(t))\| \leq \gamma_{0i} \left\|(\tilde{\eta}_i(t_0), \tilde{X}_i(t_0))\right\| e^{-\sigma_{0i}(t-t_0)}, \ \forall t \geq t_0 \geq 0 \tag{4.30}$$

where γ_{0i} is a positive constant. That is we can use (4.19) to reconstruct $\hat{\vartheta}_i$, an estimate of $\hat{X}_i$ from (4.22).

4.3.2 Un-damped case

Note that in this $D_{\eta i}(\eta_i) = 0$ since $D_i = 0$. Hence the time derivative of (4.28) along the solution of (4.21) and using (4.20) in this case satisfies

$$\dot{V}_{01i} = -\tilde{\eta}_i^T Q_{01i} \tilde{\eta}_i \leq 0 \tag{4.31}$$

which implies $\left\|(\tilde{\eta}_i(t), \tilde{X}_i(t))\right\| \leq \delta_{0i}(\cdot)$ with $\delta_{0i}(\cdot)$ being a non-decreasing function of $\left\|(\tilde{\eta}_i(t_0), \tilde{X}_i(t_0))\right\|$. Next, we consider the Lyapunov function

$$V_{02i} = \mu_{01i}(\tilde{\eta}_i^T P_{01i} \tilde{\eta}_i + \tilde{X}_i^T P_{02i} \tilde{X}_i) - \mu_{02i} \tilde{\eta}_i^T G_i(\eta_i) \tilde{X}_i \tag{4.32}$$

where μ_{01i} and μ_{02i} are positive constants to be picked such that V_{02i} is positive definite. Since $G_i(\eta_i) = \bar{J}_i(\eta_i) Q_i^{-1}(\eta_i)$, we have

$$\delta_{02im} \left\|(\tilde{\eta}_i, \tilde{X}_i)\right\|^2 \leq V_{02i} \leq \delta_{02iM} \left\|(\tilde{\eta}_i, \tilde{X}_i)\right\|^2 \tag{4.33}$$

where

$$\delta_{02im} = \mu_{01i} \min\left(\lambda_m(P_{01i}), \lambda_m(P_{02i})\right) - \mu_{02i} \max\left(1, 1/(b_i \Delta_i (m_{11i} - m_{12i}))\right),$$
$$\delta_{02iM} = \mu_{01i} \max\left(\lambda_m(P_{01i}), \lambda_m(P_{02i})\right) + \mu_{02i} \max\left(1, 1/(b_i \Delta_i (m_{11i} - m_{12i}))\right).$$

Hence V_{02} is positive definite if μ_{01i} and μ_{02i} are picked, for chosen P_{01i} and P_{02i}, such that

$$\delta_{02im} \geq \delta_{02im}^* > 0. \tag{4.34}$$

By taking derivative of (4.32) along the solution of (4.21) and noting that $D_{\eta i}(\eta_i) = 0$, we arrive at

$$\begin{aligned} \dot{V}_{02i} = & -\mu_{01i} \tilde{\eta}_i^T Q_{01i} \tilde{\eta}_i - \mu_{02i} \tilde{X}_i^T G_i(\eta_i)^T G_i(\eta_i) \tilde{X}_i - \mu_{02i} \tilde{\eta}_i^T K_{01i}^T G_i(\eta_i) \tilde{X}_i + \\ & \mu_{02i} \tilde{\eta}_i^T G_i(\eta_i)^T P_{02i}^{-1} G_i(\eta_i) P_{01i} \tilde{X}_i - \mu_{02i} \tilde{\eta}_i^T \dot{G}_i(\eta_i) \tilde{X}_i. \end{aligned} \tag{4.35}$$

After some bounding calculation, we have

$$\begin{aligned} & -\tilde{X}_i^T G_i(\eta_i)^T G_i(\eta_i) \tilde{X}_i \leq -A_{1i} \left\|\tilde{X}_i\right\|^2, \\ & -\tilde{\eta}_i^T K_{01i}^T G_i(\eta_i) \tilde{X}_i \leq A_{2i} \left(1/(4\xi_{01i}) \|\tilde{\eta}_i\|^2 + \xi_{01i} \left\|\tilde{X}_i\right\|^2\right), \\ & \tilde{\eta}_i^T G_i(\eta_i)^T P_{02i}^{-1} G_i(\eta_i) P_{01i} \tilde{X}_i \leq A_{3i} \|\tilde{\eta}_i\|^2, \\ & -\tilde{\eta}_i^T \dot{G}_i(\eta_i) \tilde{X}_i = -\tilde{\eta}_i^T G_{\eta i}(\eta_i) \dot{\phi}_i \tilde{X}_i \leq A_{4i} w_{Mi} \left(1/(4\xi_{01i}) \|\tilde{\eta}_i\|^2 + \xi_{01i} \left\|\tilde{X}_i\right\|^2\right) \end{aligned} \tag{4.36}$$

with

$$\begin{aligned} A_{1i} &:= \min\left(1, 1/(b_i \Delta_i (m_{11i} - m_{12i}))^2\right), \\ A_{2i} &:= \lambda_m(K_{01i}) \max\left(1, 1/(b_i \Delta_i (m_{11i} - m_{12i}))\right), \\ A_{3i} &:= \lambda_M(P_{01i})/\lambda_m(P_{02i}) \max\left(1, 1/(b_i \Delta_i (m_{11i} - m_{12i}))^2\right), \\ A_{4i} &:= \max\left(1 + c_i \Delta_i, c_i/(b_i (m_{11i} - m_{12i}))\right) \end{aligned}$$

where ξ_{01i} is a positive constant. Substituting (4.36) into (4.35) yields

$$\begin{aligned}\dot{V}_{02i} \leq & -\left(\mu_{01i}\lambda_m(Q_{01i}) - \mu_{02i}\left(1/(4\xi_{01i})(A_{2i} + A_{4i}w_{Mi}) + A_{3i}\right)\right)\|\tilde{\eta}_i\|^2 - \\ & \mu_{02i}\left(A_{1i} - \xi_{01i}(A_{2i} + A_{4i}w_{Mi})\right)\left\|\tilde{X}_i\right\|^2. \end{aligned} \quad (4.37)$$

It is seen from (4.37) that we can pick small constants ξ_{01i}, μ_{01i} and a large enough constant μ_{02i} such that (4.34) holds and

$$\dot{V}_{02i} \leq -\xi_{02i}\|\tilde{\eta}_i\|^2 - \xi_{03i}\left\|\tilde{X}_i\right\|^2 \quad (4.38)$$

where ξ_{02i} and ξ_{03i} are positive constants, which in turn implies global asymptotic stability of (4.21) in the un-damped case. Note that (4.38) in general does not imply global exponential stability of (4.21) since w_{Mi} can depend on the initial conditions. □

4.4 Path tracking error dynamics and control design

In this section, we will consider the dynamics (4.25) and design controls τ_{vci} and τ_{wci} for the robot i to achieve the formation control objective posed in Section 2. The actual control vector τ_i is then calculated from (4.27). The main ideas of the control design are as follows:

1) Tracking errors of the robot i are interpreted in a frame attached to the reference path $\Gamma_i(s_i)$ such that the error dynamics are of a triangular form to which the backstepping technique [12] can be applied.

2) The orientation error and estimate of the robot linear velocity are used as virtual controls to stabilize the tangential and cross-track errors interpreted in the frame attached to the robot reference path. The reason that the orientation error is not directly stabilized at this stage but is treated as a virtual control to stabilize the cross-track error, is to overcome difficulties caused by the nonholonomic feature of the robot.

3) The estimate of the robot angular velocity is then used as a virtual control to stabilize the orientation error interpreted in the frame attached to frame attached to the robot reference path.

4) The controls τ_{vci} and τ_{wci} are finally designed to force the actual estimates of the robot velocities to converge to their virtual values by using the backstepping technique.

5) The above control designs are carried out in such a way that the derivative of the path parameter $\dot{s}_i$ for the robot i is saved as an additional control input. This control input is then used to synchronize all the path parameters. The path parameter derivative, $\dot{s}_0$, of the path of the virtual structure center will be chosen as a function of the path tracking errors of the robots and time. This choice means that the control system takes mobility of each robot into account, i.e. a formation feedback is exhibited in the control system.

4.4.1 Path tracking error dynamics

To prepare for the control design, we first interpret the path tracking errors $((x_i - x_{di}), (y_i - y_{di}), (\phi_i - \phi_{di}))$ in the frame $O_{bi}X_{bi}Y_{bi}$, of which the origin O_{bi} is a point on the path $\Gamma_i(s_i)$, $O_{bi}X_{bi}$ and $O_{bi}Y_{bi}$ axes are parallel to the surge axis and sway axis of the robot, respectively, see Figure 4.2. In Figure 4.2, the direction of v_{di} is tangential to the path $\Gamma_i(s_i)$. From this figure, we have:

$$\begin{aligned} x_{ei} &= (x_i - x_{di})\cos(\phi_i) + (y_i - y_{di})\sin(\phi_i), \\ y_{ei} &= -(x_i - x_{di})\sin(\phi_i) + (y_i - y_{di})\cos(\phi_i), \\ \phi_{ei} &= \phi_i - \phi_{di}. \end{aligned} \tag{4.39}$$

The physical interpretations of the new tracking errors $(x_{ei}, y_{ei}, \phi_{ei})$ defined in (4.39) are: x_{ei} is the tangential tracking error, y_{ei} is the cross-tracking error, and ϕ_{ei} is the heading error. Differentiating both sides of (4.39) along the solution of (4.25), we have

$$\begin{aligned} \dot{x}_{ei} &= \hat{v}_i - v_{di}\cos(\phi_{ei}) + y_{ei}(\hat{w}_i + \tilde{w}_i) + \tilde{v}_i \\ \dot{y}_{ei} &= v_{di}\sin(\phi_{ei}) - x_{ei}(\hat{w}_i + \tilde{w}_i) \\ \dot{\phi}_{ei} &= \hat{w}_i - w_{di} + \tilde{w}_i \\ \dot{\hat{v}}_i &= \tau_{vci} + \Omega_{vi} \\ \dot{\hat{w}}_i &= \tau_{wci} + \Omega_{wi} \end{aligned} \tag{4.40}$$

where we have defined

$$v_{di} = \sqrt{x_{di}'^2(s_i) + y_{di}'^2(s_i)}\,\dot{s}_i, \quad w_{di} = \frac{x_{di}'(s_i)y_{di}''(s_i) - x_{di}''(s_i)y_{di}'(s_i)}{x_{di}'^2(s_i) + y_{di}'^2(s_i)}\dot{s}_i. \tag{4.41}$$

In the above expressions, we refer v_{di} and w_{di} to as the desired linear and angular velocities of the robot on the path. It is noted that speed of the robot on the path is specified by $\dot{s}_i$, and that (4.41) is well defined, see Assumption 4.1. From (4.39) for any value of ϕ_i, $(x_{ei}, y_{ei}, \phi_{ei}) = 0$ if and only if $(x_i, y_i, \phi_i) = (x_{di}, y_{di}, \phi_{di})$, we have therefore converted the problem of output-feedback path tracking for the mobile robot in question to a problem of stabilizing (4.40) at the origin.

4.4.2 Control design

It is seen that (4.40) is of the lower triangular structure with respect to $\hat{v}_i$ and $\hat{w}_i$, we use the backstepping technique [12] to design τ_{vci} and τ_{wci} in three steps.

Step 1: Stabilizing the (x_{ei}, y_{ei}) dynamics. In this step, we consider $\hat{v}_i$ and ϕ_{ei} as controls. Define

$$v_{ei} = \hat{v}_i - \alpha_{vi},\ \bar{\phi}_{ei} = \phi_{ei} - \alpha_{\phi_{ei}} \tag{4.42}$$

where α_{vi} and $\alpha_{\phi_{ei}}$ are the virtual controls of $\hat{v}_i$ and ϕ_{ei}, respectively. Moreover, we define

$$\dot{\tilde{s}}_i = \dot{s}_i - \omega_i(t, x_e, y_e, \phi_e) \tag{4.43}$$

where $x_e = [x_{e1}, x_{e2}, ..., x_{er}]^T$, $y_e = [y_{e1}, y_{e2}, ..., y_{er}]^T$, $\phi_e = [\phi_{e1}, \phi_{e2}, ..., \phi_{er}]^T$; $\omega_i(t, x_e, y_e, \phi_e)$ is a strictly positive function that specifies how fast the i^{th} mobile robot should move to maintain the formation since $\dot{s}_i$ is related to the desired forward speed, see (4.41). This function will be specified later. Substituting (4.42) and (4.43) into the first two equations of (4.40), and choosing the virtual controls α_{vi} and $\alpha_{\phi_{ei}}$ as

$$\alpha_{vi} = -\frac{k_{1i}x_{ei}}{\Delta_{1i}} + \bar{v}_{di}\omega_i \cos(\phi_{ei}),\ \alpha_{\phi_{ei}} = -\arctan(\frac{k_{2i}y_{ei}}{\Delta_{1i}}) \tag{4.44}$$

where $\Delta_{1i} = \sqrt{1 + x_{ei}^2 + y_{ei}^2}$, $\bar{v}_{di} = \sqrt{x_{di}'^2(s_i) + y_{di}'^2(s_i)}$; k_{1i} and k_{2i} are positive constants, result in the (x_{ei}, y_{ei}) error dynamics

$$\begin{aligned}
\dot{x}_{ei} &= -\frac{k_{1i}x_{ei}}{\Delta_{1i}} + v_{ei} - \bar{v}_{di}\dot{\tilde{s}}_i \cos(\phi_{ei}) + y_{ei}(\hat{w}_i + \tilde{w}_i) + \tilde{v}_i,\\
\dot{y}_{ei} &= -\frac{k_{2i}\bar{v}_{di}\omega_i y_{ei}}{\Delta_{2i}} - \frac{k_{2i}\bar{v}_{di}\dot{\tilde{s}}_i y_{ei}}{\Delta_{2i}} + \bar{v}_{di}(\dot{\tilde{s}}_i + \omega_i)(\sin(\bar{\phi}_{ei})\cos(\alpha_{\phi_{ei}}) +\\
&\quad (\cos(\bar{\phi}_{ei}) - 1)\sin(\alpha_{\phi_{ei}})) - x_{ei}(\hat{w}_i + \tilde{w}_i)
\end{aligned} \tag{4.45}$$

where $\Delta_{2i} = \sqrt{1 + x_{ei}^2 + (1 + k_{2i}^2)y_{ei}^2}$.

Step 2: Stabilizing the $\bar{\phi}_{ei}$ dynamics. In this step, we consider $\hat{w}_i$ as a control. Define

$$w_{ei} = \hat{w}_i - \alpha_{wi} \tag{4.46}$$

where α_{wi} is the virtual control of $\hat{w}_i$. From (4.42) and (4.44), the first time derivative of $\bar{\phi}_{ei}$ is given by

$$\dot{\bar{\phi}}_{ei} = \left(1 - k_{2i}x_{ei}\frac{\Delta_{1i}}{\Delta_{2i}^2}\right)(\alpha_{wi} + w_{ei} + \tilde{w}_i) + \frac{k_{2i}}{\Delta_{1i}\Delta_{2i}^2}(\bar{v}_{di}\omega_i \sin(\phi_{ei})(1 + x_{ei}^2) - x_{ei}y_{ei}(\alpha_{vi} - \bar{v}_{di}\omega_i\cos(\phi_{ei}))) + \frac{k_{2i}\bar{v}_{di}\dot{\bar{s}}_i}{\Delta_{1i}\Delta_{2i}^2}(\bar{v}_{di}\omega_i \sin(\phi_{ei})(1 + x_{ei}^2) + x_{ei}y_{ei}\bar{v}_{di}\omega_i\cos(\phi_{ei})) - \frac{k_{2i}x_{ei}y_{ei}}{\Delta_{1i}\Delta_{2i}^2}(v_{ei} + \tilde{v}_i) - \bar{w}_{di}(\omega_i + \dot{\bar{s}}_i). \quad (4.47)$$

If the constant k_{2i} is chosen such that $0 < k_{2i} < 1$, then from (4.47) the virtual control α_{wi} is selected as

$$\alpha_{wi} = \frac{1}{(1 - k_{2i}x_{ei}\Delta_{1i}\Delta_{2i}^{-2})}\left(-\frac{k_{3i}\bar{\phi}_{ei}}{\Delta_{3i}} + \bar{w}_{di}\omega_i - \frac{k_{2i}}{\Delta_{1i}\Delta_{2i}^2}(\bar{v}_{di}\omega_i\sin(\phi_{ei}) \times (1 + x_{ei}^2) - x_{ei}y_{ei}(\alpha_{vi} - \bar{v}_{di}\omega_i\cos(\phi_{ei}))) - \frac{\bar{v}_{di}\omega_i y_{ei}}{\Delta_{1i}\bar{\phi}_{ei}}(\sin(\bar{\phi}_{ei})\cos(\alpha_{\phi_{ei}}) + (\cos(\bar{\phi}_{ei}) - 1)\sin(\alpha_{\phi_{ei}}))\right) \quad (4.48)$$

where $\Delta_{3i} = \sqrt{1 + \bar{\phi}_{ei}^2}$, $\bar{w}_{di} = \frac{(x'_{di}(s_i)y''_{di}(s_i) - x''_{di}(s_i)y'_{di}(s_i))}{(x'^2_{di}(s_i) + y'^2_{di}(s_i))}$, k_{3i} is a positive constant. The last term in (4.48) is included to cancel the cross term in the y_{ei}-dynamics. It is noted that (4.48) is well defined since $1 - k_{2i}x_{ei}\Delta_{1i}\Delta_{2i}^{-2} \geq 1 - k_{2i} > 0$; $\sin(\bar{\phi}_e)/\bar{\phi}_e = \int_0^1 \cos(\bar{\phi}_e\lambda)d\lambda$ and $(\cos(\bar{\phi}_e) - 1)/\bar{\phi}_e = \int_0^1 \sin(\bar{\phi}_e\lambda)d\lambda$ are smooth functions of $\bar{\phi}_e$. It is also of interest to note that the upper bound of α_{wi} is given by

$$|\alpha_{wi}| \leq \frac{k_{3i} + |\bar{w}_{di}|\omega_i + k_{2i}(3|\bar{v}_{di}|\omega_i + k_{1i}) + 2|\bar{v}_{di}|\omega_i}{1 - k_{2i}} := \alpha_{wi}^M. \quad (4.49)$$

Substituting (4.48) into (4.47) yields

$$\dot{\bar{\phi}}_{ei} = -\frac{k_{3i}\bar{\phi}_{ei}}{\Delta_{3i}} - \frac{\bar{v}_{di}\omega_i y_{ei}}{\Delta_{1i}\bar{\phi}_{ei}}(\sin(\bar{\phi}_{ei})\cos(\alpha_{\phi_{ei}}) + (\cos(\bar{\phi}_{ei}) - 1)\sin(\alpha_{\phi_{ei}})) + (1 - k_{2i}x_{ei}\Delta_{1i}\Delta_{2i}^{-2})(w_{ei} + \tilde{w}_i) + \frac{k_{2i}\bar{v}_{di}\dot{\bar{s}}_i}{\Delta_{1i}\Delta_{2i}^2}(\bar{v}_{di}\omega_i\sin(\phi_{ei})(1 + x_{ei}^2) + x_{ei}y_{ei}\bar{v}_{di}\omega_i\cos(\phi_{ei})) - \frac{k_{2i}x_{ei}y_{ei}}{\Delta_{1i}\Delta_{2i}^2}(v_{ei} + \tilde{v}_i) - \bar{w}_{di}\dot{\bar{s}}_i. \quad (4.50)$$

Step 3: Stabilizing the (v_{ei}, w_{ei}) dynamics. In this step, the controls τ_{vci} and τ_{wci} are designed. From (4.42)and (4.44), the time derivative of v_{ei} is given by

$$\begin{aligned}\dot{v}_{ei} = & \tau_{vci} + \Omega_{vi} - \frac{\partial \alpha_{vi}}{\partial s_i}(\dot{\hat{s}}_i + \omega_i) - \frac{\partial \alpha_{vi}}{\partial t} - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial x_{ej}} \left(\hat{v}_j - \bar{v}_{dj}(\dot{\hat{s}}_j + \omega_j) \times \cos(\phi_{ej}) + y_{ej}(\hat{w}_j + \tilde{w}_j) + \tilde{v}_j\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial y_{ej}} \left(\bar{v}_{dj}(\dot{\hat{s}}_j + \omega_j)\sin(\phi_{ej}) - x_{ej}(\hat{w}_j + \tilde{w}_j)\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial \phi_{ej}} \left(\hat{w}_j - \bar{w}_{dj}(\dot{\hat{s}}_j + \omega_j) + \tilde{w}_j)\right). \end{aligned} \tag{4.51}$$

From (4.51), we choose the control τ_{vci} as

$$\begin{aligned}\tau_{vci} = & -k_{4i}v_{ei} + \frac{\partial \alpha_{vi}}{\partial s_i}\omega_i + \frac{\partial \alpha_{vi}}{\partial t} + \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial x_{ej}} \left(\hat{v}_j - \bar{v}_{dj}\omega_j \cos(\phi_{ej}) + y_{ej}\hat{w}_j\right) + \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial y_{ej}} \left(\bar{v}_{dj}\omega_j \sin(\phi_{ej}) - x_{ej}\hat{w}_j\right) + \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial \phi_{ej}} \left(\hat{w}_j - \bar{w}_{dj}\omega_j\right) - \frac{x_{ei}}{\Delta_{1i}} + \frac{k_{2i}x_{ei}y_{ei}\bar{\phi}_{ei}}{\Delta_{1i}\Delta_{2i}^2} \end{aligned} \tag{4.52}$$

where k_{4i} is a positive constant. The last two terms in (4.52) are included to cancel the cross terms in x_{ei} and $\bar{\phi}_{ei}$ dynamics. Substituting (4.52) into (4.51) gives

$$\begin{aligned}\dot{v}_{ei} = & -k_{4i}v_{ei} + \Omega_{vi} - \frac{\partial \alpha_{vi}}{\partial s_i}\dot{\hat{s}}_i - \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial x_{ej}} \left(-\bar{v}_{dj}\dot{\hat{s}}_j \cos(\phi_{ej}) + y_{ej}\tilde{w}_j + \tilde{v}_j\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial y_{ej}} \left(\bar{v}_{dj}\dot{\hat{s}}_j \sin(\phi_{ej}) - x_{ej}\tilde{w}_j\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial \phi_{ej}} \left(-\bar{w}_{dj}\dot{\hat{s}}_j + \tilde{w}_j)\right) - \frac{x_{ei}}{\Delta_{1i}} + \frac{k_{2i}x_{ei}y_{ei}\bar{\phi}_{ei}}{\Delta_{1i}\Delta_{2i}^2}. \end{aligned} \tag{4.53}$$

From (4.46) and (4.48), the time derivative of w_{ei} is given by

$$\begin{aligned}\dot{w}_{ei} = \; & \tau_{wci} + \Omega_{wi} - \frac{\partial \alpha_{wi}}{\partial s_i}(\dot{\hat{s}}_i + \omega_i) - \frac{\partial \alpha_{wi}}{\partial t} - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial x_{ej}} \left(\hat{v}_j - \bar{v}_{dj}(\dot{\hat{s}}_j + \omega_j)\cos(\phi_{ej}) + y_{ej}(\hat{w}_j + \tilde{w}_j) + \tilde{v}_j\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{vi}}{\partial w_{ej}} \left(\bar{v}_{dj}(\dot{\hat{s}}_j + \omega_j)\sin(\phi_{ej}) - x_{ej}(\hat{w}_j + \tilde{w}_j)\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial \phi_{ej}} \left(\hat{w}_j - \bar{w}_{dj}(\dot{\hat{s}}_j + \omega_j) + \tilde{w}_j)\right). \end{aligned} \tag{4.54}$$

From (4.54), the control τ_{wci} is chosen as

$$\begin{aligned}\tau_{wci} = \; & -k_{5i}w_{ei} + \frac{\partial \alpha_{wi}}{\partial s_i}\omega_i + \frac{\partial \alpha_{wi}}{\partial t} + \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial x_{ej}} \left(\hat{v}_j - \bar{v}_{dj}\omega_j\cos(\phi_{ej}) + y_{ej}\hat{w}_j\right) + \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial y_{ej}} \left(\bar{v}_{dj}\omega_j\sin(\phi_{ej}) - x_{ej}\hat{w}_j\right) + \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial \phi_{ej}} \left(\hat{w}_j - \bar{w}_{dj}\omega_j\right) - \left(1 - k_{2i}x_{ei}\Delta_{1i}\Delta_{2i}^{-2}\right)\bar{\phi}_{ei} \end{aligned} \tag{4.55}$$

where k_{5i} is a positive constant. Substituting (4.55) into (4.54) yields

$$\begin{aligned}\dot{w}_{ei} = \; & -k_{5i}w_{ei} + \Omega_{wi} - \frac{\partial \alpha_{wi}}{\partial s_i}\dot{\hat{s}}_i - \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial x_{ej}} \left(-\bar{v}_{dj}\dot{\hat{s}}_j\cos(\phi_{ej}) + y_{ej}\tilde{w}_j + \tilde{v}_j\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial y_{ej}} \left(\bar{v}_{dj}\dot{\hat{s}}_j\sin(\phi_{ej}) - x_{ej}\tilde{w}_j\right) - \\ & \sum_{j=1}^{N} \frac{\partial \alpha_{wi}}{\partial \phi_{ej}} \left(-\bar{w}_{dj}\dot{\hat{s}}_j + \tilde{w}_j)\right) - \left(1 - k_{2i}x_{ei}\Delta_{1i}\Delta_{2i}^{-2}\right)\bar{\phi}_{ei}. \end{aligned} \tag{4.56}$$

We now need to choose all the functions $\omega_i(t, x_e, y_e, \phi_e)$ and the update law for $\dot{\hat{s}}_i$ so that (46) holds. To do this, we consider the following Lyapunov function

$$V_1 = \sum_{i=1}^{N} \left(\sqrt{1 + x_{ei}^2 + y_{ei}^2} + \sqrt{1 + \bar{\phi}_{ei}^2} - 2 + 0.5v_{ei}^2 + 0.5w_{ei}^2 + 0.5\gamma_i(s_i - s_0)^2\right) \tag{4.57}$$

where γ_i is a positive constant. We note that the choice of weighting on the path parameters in (4.57) is not unique. The time derivative of (4.57) along the solutions of (4.45), (4.50), (4.53) and (4.56) is

$$\dot{V}_1 = -\sum_{i=1}^{N}\left(\frac{k_{1i}x_{ei}^2}{\Delta_{1i}^2} + \frac{k_{2i}\bar{v}_{di}\omega_i y_{ei}^2}{\Delta_{1i}\Delta_{2i}^2} + \frac{k_{3i}\bar{\phi}_{ei}^2}{\Delta_{3i}^2} + k_{4i}v_{ei}^2 + k_{5i}w_{ei}^2\right) + \sum_{i=1}^{N}\Phi_{1i} + \sum_{i=1}^{N}\Phi_{21i}\dot{\bar{s}}_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\Phi_{22ij}\dot{\bar{s}}_j + \sum_{i=1}^{N}\gamma_i(s_i - s_0)(\dot{\bar{s}}_i + \omega_i - \dot{s}_0) \quad (4.58)$$

where

$$\Phi_{1i} = \frac{x_{ei}}{\Delta_{1i}}\tilde{v}_i + \left(1 - k_{2i}x_{ei}\Delta_{1i}\Delta_{2i}^{-2}\right)\Delta_{3i}^{-1}\tilde{w}_i\bar{\phi}_{ei} - \frac{k_{2i}x_{ei}y_{ei}}{\Delta_{1i}\Delta_{2i}^2\Delta_{3i}}\tilde{v}_i\bar{\phi}_{ei} + \Omega_{vi}v_{ei} + \Omega_{wi}w_{ei} - \sum_{j=1}^{N}\left(\frac{\partial\alpha_{vi}}{\partial x_{ej}}\left(y_{ej}\tilde{w}_j + \tilde{v}_j\right)v_{ei} - \frac{\partial\alpha_{vi}}{\partial y_{ej}}x_{ej}\tilde{w}_j v_{ei} - \frac{\partial\alpha_{vi}}{\partial\phi_{ej}}\tilde{w}_j v_{ei} + \frac{\partial\alpha_{wi}}{\partial x_{ej}}\left(y_{ej}\tilde{w}_j + \tilde{v}_j\right)w_{ei} - \frac{\partial\alpha_{wi}}{\partial y_{ej}}x_{ej}\tilde{w}_j w_{ei} - \frac{\partial\alpha_{wi}}{\partial\phi_{ej}}\tilde{w}_j w_{ei}\right) \quad (4.59)$$

$$\Phi_{21i} = -\frac{x_{ei}}{\Delta_{1i}}\bar{v}_{di}\cos(\phi_{ei}) - \frac{k_{2i}\bar{v}_{di}y_{ei}}{\Delta_{1i}\Delta_{2i}} + \frac{\bar{v}_{di}y_{ei}}{\Delta_{1i}}(\sin(\bar{\phi}_{ei})\cos(\alpha_{\phi_{ei}}) + (\cos(\bar{\phi}_{ei}) - 1)\sin(\alpha_{\phi_{ei}})) + \frac{k_{2i}\bar{v}_{di}}{\Delta_{1i}\Delta_{2i}^2\Delta_{3i}}(\bar{v}_{di}\omega_i\sin(\phi_{ei})(1 + x_{ei}^2) + x_{ei}y_{ei}\bar{v}_{di}\omega_i\cos(\phi_{ei}))\bar{\phi}_{ei} \quad (4.60)$$

$$\Phi_{22ij} = \sum_{j=1}^{N}\left(-\frac{\partial\alpha_{vi}}{\partial x_{ej}}\bar{v}_{dj}\cos(\phi_{ej})v_{ei} + \frac{\partial\alpha_{vi}}{\partial y_{ej}}\bar{v}_{dj}\sin(\phi_{ej})v_{ei} - \frac{\partial\alpha_{vi}}{\partial\phi_{ej}}\bar{w}_{dj}v_{ei} - \frac{\partial\alpha_{wi}}{\partial x_{ej}}\bar{v}_{dj}\cos(\phi_{ej})v_{ei} + \frac{\partial\alpha_{wi}}{\partial y_{ej}}\bar{v}_{dj}\sin(\phi_{ej})v_{ei} - \frac{\partial\alpha_{wi}}{\partial\phi_{ej}}\bar{w}_{dj}v_{ei}\right) \quad (4.61)$$

By noting that $\sum_{i=1}^{N}\sum_{j=1}^{N}\Phi_{22ij}\dot{\bar{s}}_j = \sum_{i=1}^{N}\sum_{j=1}^{N}\Phi_{22ji}\dot{\bar{s}}_i$, we can write (4.58) as

$$\dot{V}_1 = -\sum_{i=1}^{N}\left(\frac{k_{1i}x_{ei}^2}{\Delta_{1i}^2} + \frac{k_{2i}\bar{v}_{di}\omega_i y_{ei}^2}{\Delta_{1i}\Delta_{2i}^2} + \frac{k_{3i}\bar{\phi}_{ei}^2}{\Delta_{3i}^2} + k_{4i}v_{ei}^2 + k_{5i}w_{ei}^2\right) + \sum_{i=1}^{N}\Phi_{1i} + \sum_{i=1}^{N}\left(\Phi_{21i} + \sum_{j=1}^{N}\Phi_{22ji}\right)\dot{\bar{s}}_i + \sum_{i=1}^{N}\gamma_i(s_i - s_0)(\dot{\bar{s}}_i + \omega_i - \dot{s}_0). \quad (4.62)$$

From (4.62), we can choose

$$\omega_i = \dot{s}_0, \quad \dot{\bar{s}}_i = -\varepsilon_s \tanh(\Phi_{21i} + \sum_{j=1}^{N} \Phi_{22ji} + \gamma_i(s_i - s_0)) \tag{4.63}$$

where ε_s is a positive constant to be selected later. We now can choose $\dot{s}_0$ in two cases: the first case is with formation feedback and the second case is without formation feedback as discussed previously.

For the case with formation feedback, $\dot{s}_0$ can be chosen as

$$\dot{s}_0 = \omega_0(t)(1-\chi_1 e^{-\chi_2(t-t_0)})(1-\chi_3 \tanh(x_e^T \Gamma_x x_e + y_e^T \Gamma_y y_e + \phi_e^T \Gamma_\phi \phi_e)) \tag{4.64}$$

where $\omega_0(t)$ is a strictly positive and bounded function. This function specifies how fast the whole group of robots should move, since the forward speed of the center of the virtual structure is given by $v_{d0} = \sqrt{x_{d0}'^2(s_0) + y_{d0}'^2(s_0)}\dot{s}_0$. The weighted positive definite matrices Γ_x, Γ_y, Γ_ϕ determining the tracking errors are taken into account in the formation feedback. All the constants χ_1, χ_2 and χ_3 are nonnegative but $\chi_1 < 1$ and $\chi_3 < 1$. If χ_1, χ_2 and χ_3 are positive, the choice of $\dot{s}_0$ in (4.64) has the following desired feature: when the tracking errors are large, the virtual structure will wait for robots; when the tracking errors converge to zero and the time increases, $\dot{s}_0$ approaches $\omega_0(t)$, i.e. the center of the virtual structure moves at the desired speed. Now we choose the constant ε_s such that

$$\varepsilon_s < \omega_0(t)(1 - \chi_1)(1 - \chi_3) \tag{4.65}$$

then we have

$$\begin{aligned} \dot{s}_i = \dot{\bar{s}}_i + \omega_i = &-\varepsilon_s \tanh(\Phi_{21i} + \sum_{j=1}^{N} \Phi_{22ji} + \gamma_i(s_i - s_0)) + \omega_0(t)(1- \\ &\chi_1 e^{-\chi_2(t-t_0)})(1 - \chi_3 \tanh(x_e^T \Gamma_x x_e + y_e^T \Gamma_y y_e + \phi_e^T \Gamma_\phi \phi_e)) \\ &\geq -\varepsilon_s + \omega_0(t)(1 - \chi_1)(1 - \chi_3) > 0. \end{aligned} \tag{4.66}$$

For the case without formation feedback, $\dot{s}_0$ can be simply chosen as

$$\dot{s}_0 = \omega_0^*(t) \tag{4.67}$$

where $\omega_0^*(t)$ is a strictly positive and bounded function, which again specifies how fast the whole group of robots should move. It is noted that in this case, the controls τ_{vci} and τ_{wci} are significantly simplified in the sense that all the terms in the sum $\sum_{i=1}^{N}$, see (4.52), (4.55) and (4.63) are zero.

Remark 4.5. In the case with formation feedback, i.e. $\dot{s}_0$ is chosen as in (4.64), each robot requires the path parameters, measurements of position and orientation of itself and all other robots in the group be available for feedback. In the case without formation feedback, i.e. $\dot{s}_0$ is chosen as in (4.67) each robot requires only the path parameter and measurements of position and orientation of itself for feedback. The trade-off is that the mobility of each robot is taken into account for the case with formation feedback but is not for the case without formation feedback. For example, if some robots get saturated or disturbed, formation cannot be achieved in the case without formation feedback.

Substituting (4.63) into (4.62) yields

$$\begin{aligned}\dot{V}_1 = & -\sum_{i=1}^{N}\left(\frac{k_{1i}x_{ei}^2}{\Delta_{1i}^2}+\frac{k_{2i}\bar{v}_{di}\omega_i y_{ei}^2}{\Delta_{1i}\Delta_{2i}^2}+\frac{k_{3i}\tilde{\phi}_{ei}^2}{\Delta_{3i}^2}+k_{4i}v_{ei}^2+k_{5i}w_{ei}^2\right)+\sum_{i=1}^{N}\Phi_{1i}- \\ & \varepsilon_s\sum_{i=1}^{N}(\Phi_{21i}+\sum_{j=1}^{N}\Phi_{22ji}+\gamma_i(s_i-s_0))\tanh(\Phi_{21i}+\sum_{j=1}^{N}\Phi_{22ji}+\gamma_i(s_i-s_0)).\end{aligned} \tag{4.68}$$

We now state the main result of this chapter in the following theorem.

Theorem 4.6. *Under Assumption 4.1, the control inputs τ_{vi} and τ_{wi} given by (4.27), (4.52) and (4.55), the observer (4.19) and (4.22), and the path parameter update law (4.63) solve the control objective (4.5) and (4.6).*

4.5 Proof of Theorem 4.6

Before prove Theorem 4.6, we note, upon tedious completion of squares, from (4.59) and (4.23) that

$$|\Phi_{1i}| \leq \varepsilon_{1i}(\|\bar{\eta}_i\|^2+\|\tilde{X}_i\|^2)+\varepsilon_{2i}(v_{ei}^2+w_{ei}^2) \tag{4.69}$$

where ε_{1i} is some (large) positive constant and ε_{2i} is an arbitrarily small positive constant. Now, to prove Theorem 4.6, we consider damped and undamped cases separately.

4.5.1 Damped case

For this case, we consider the following Lyapunov function candidate

$$V_{21} = V_1 + \mu_{03}\sum_{i=1}^{N} V_{01i} \tag{4.70}$$

where μ_{03} is a large enough positive constant to be specified later, and V_{01i} is given in (4.28). Using (4.29) and (74), the derivative of (4.70), satisfies

$$\dot{V}_{21} \leq -\sum_{i=1}^{N} \left(\frac{k_{1i} x_{ei}^2}{\Delta_{1i}^2} + \frac{k_{2i} \bar{v}_{di} \omega_i y_{ei}^2}{\Delta_{1i} \Delta_{2i}^2} + \frac{k_{3i} \bar{\phi}_{ei}^2}{\Delta_{3i}^2} + (k_{4i} - \varepsilon_{2i}) v_{ei}^2 + (k_{5i} - \varepsilon_{2i}) w_{ei}^2 \right) -$$
$$\varepsilon_s \sum_{i=1}^{N} (\Phi_{21i} + \sum_{j=1}^{N} \Phi_{22ji} + \gamma_i (s_i - s_0)) \tanh(\Phi_{21i} + \sum_{j=1}^{N} \Phi_{22ji} + \gamma_i (s_i - s_0)) -$$
$$\sum_{i=1}^{N} (\mu_{03} \lambda_m (Q_{01i}) - \varepsilon_{1i}) \|\tilde{\eta}_i\|^2 - (\mu_{03} \lambda_m (Q_{02i}) - \varepsilon_{1i}) \left\| \tilde{X}_i \right\|^2. \tag{4.71}$$

We can now pick the constants μ_{03} large enough and ε_{2i} small enough such that $k_{4i} - \varepsilon_{2i} > 0$, $k_{5i} - \varepsilon_{2i} > 0$, $\mu_{03} \lambda_m (Q_{01i}) - \varepsilon_{1i} > 0$, $\mu_{03} \lambda_m (Q_{02i}) - \varepsilon_{1i} > 0$. Then the inequality of (4.71) means that $\dot{V}_{21}(t) \leq 0$, i.e. ($x_{ei}(t)$, $y_{ei}(t)$, $\phi_{ei}(t)$, $v_{ei}(t)$, $w_{ei}(t)$, $\tilde{\eta}_i(t)$, $\tilde{X}_i(t)$, $s_i(t) - s_0(t)$), is bounded. Applying Babarlat's lemma to (76) results in

$$\lim_{t \to \infty} (x_{ei}(t), y_{ei}(t), \phi_{ei}(t), v_{ei}(t), w_{ei}(t), \tilde{\eta}_i(t), \tilde{X}_i(t)) = 0$$
$$\lim_{t \to \infty} (\sum_{i=1}^{N} (\Phi_{21i}(t) + \sum_{j=1}^{N} \Phi_{22ji}(t) + \gamma_i (s_i(t) - s_0(t))) \tanh(\Phi_{21i}(t) +$$
$$\sum_{j=1}^{N} \Phi_{22ji}(t) + \gamma_i (s_i(t) - s_0(t))) = 0. \tag{4.72}$$

On the other hand, from the first equation of (4.72), (4.60) and (78), we have

$$\lim_{t \to \infty} (\Phi_{21i}(t) + \sum_{j=1}^{N} \Phi_{22ji}(t)) = 0. \tag{4.73}$$

From this fact, the second equation of (4.72) implies that

$$\lim_{t \to \infty} (s_i(t) - s_0(t))) = 0 \tag{4.74}$$

which completes the proof of Theorem 4.6 for the damped case.

4.5.2 Un-damped case.

In this case, we consider the Lyapunov function

$$V_{22} = \mu_{04} V_1 + \sum_{i=1}^{N} V_{02i} \tag{4.75}$$

where V_{02i} is given in (4.32), and μ_{04} is a positive constant to be picked later. We remind the reader that here we do not assume $w_i(t)$ is bounded by w_{Mi} as in the proof of Theorem 4.3. To prepare for calculating the derivative of

(4.75), we here do not use the bound of $-\tilde{\eta}_i^T \dot{G}_i(\eta_i)\tilde{X}_i$ as in the last inequality of (4.36). Instead, we proceed as follows

$$\begin{aligned}
&-\tilde{\eta}_i^T \dot{G}_i(\eta_i)\tilde{X}_i = -\tilde{\eta}_i^T G_{\eta i}(\eta_i)\dot{\phi}_i\tilde{X}_i = -\tilde{\eta}_i^T G_{\eta i}(\eta_i)(\hat{w}_i + \tilde{w}_i)\tilde{X}_i \\
&= -\tilde{\eta}_i^T G_{\eta i}(\eta_i)(\alpha_{wi} + w_{ei} + \tilde{w}_i)\tilde{X}_i \\
&= -\tilde{\eta}_i^T G_{\eta i}(\eta_i)(\alpha_{wi} + \tilde{w}_i)\tilde{X}_i - \tilde{\eta}_i^T G_{\eta i}(\eta_i) w_{ei}\tilde{X}_i \\
&\leq A_{4i}(\alpha_{wi}^M + \gamma_{0i}\delta_{0i}(\bullet))((1/4\xi_{01i})\|\tilde{\eta}_i\|^2 + \xi_{01i}\left\|\tilde{X}_i\right\|^2) + \\
&\mu_{04}\varepsilon_{3i} w_{ei}^2 + 1/(4\mu_{04}\varepsilon_{3i})A_{4i}^2\delta_{0i}^2(\bullet)\|\tilde{\eta}_i\|^2
\end{aligned} \tag{4.76}$$

where α_{wi}^M is defined in (4.49), and ε_{3i} is a positive constant to be picked later. By using the first three inequalities of (4.36) and (4.76), differentiating both sides of (4.75) yields

$$\begin{aligned}
\dot{V}_{22} \leq\ & -\mu_{04}\sum_{i=1}^{N}\left(\frac{k_{1i}x_{ei}^2}{\Delta_{1i}^2} + \frac{k_{2i}\bar{v}_{di}\omega_i y_{ei}^2}{\Delta_{1i}\Delta_{2i}^2} + \frac{k_{3i}\bar{\phi}_{ei}^2}{\Delta_{3i}^2} + k_{4i}v_{ei}^2 + (k_{5i} - \varepsilon_{3i})w_{ei}^2\right) - \\
& \varepsilon_s\sum_{i=1}^{N}(\Phi_{21i} + \sum_{j=1}^{N}\Phi_{22ji} + \gamma_i(s_i - s_0))\tanh(\Phi_{21i} + \sum_{j=1}^{N}\Phi_{22ji} + \\
& \gamma_i(s_i - s_0)) - \sum_{i=1}^{N}(k_{6i}\|\tilde{\eta}_i\|^2 + k_{7i}\left\|\tilde{X}_i\right\|^2)
\end{aligned} \tag{4.77}$$

where

$$\begin{aligned}
k_{6i} &= \mu_{01i}\lambda_m(Q_{01i}) - \mu_{02i}(1/(4\xi_{01i})A_{2i} + A_{4i}(\alpha_{wi}^M + \\
&\quad \gamma_{0i}\delta_{0i}(\bullet))((1/4\xi_{01i})) + A_{3i}) - 1/(4\mu_{04}\varepsilon_{3i})A_{4i}^2\delta_{0i}^2(\bullet), \\
k_{7i} &= \mu_{02i}(A_{1i} - \xi_{01i}(A_{2i} + A_{4i}\xi_{01i}))
\end{aligned} \tag{4.78}$$

A closed look at (4.78) shows that we can pick small enough ε_{3i}, μ_{04}, ξ_{01i} and large enough μ_{01i} such that

$$k_{6i} \geq k_{6i}^*, \quad k_{7i} \geq k_{7i}^* \tag{4.79}$$

where k_i^*, $i = 6, 7$ are strictly positive constants. From (4.77) and (4.79), using the same arguments as in the damped case, we complete the proof of Theorem 4.6. Finally we note that all of the constants μ_{01i}, μ_{02i}, μ_{03}, μ_{04}, ξ_{01i}, ε_{1i}, ε_{2i}, ε_{3i} are used only for the proof. We do not need to choose them to implement the proposed controller.

4.6 Simulations

In this section we simulate formation control of a group of three identical mobile robots to illustrate the effectiveness of the proposed controller. The physical parameters are taken from Section 1.3.1, Chapter 1. The observer and control gains, the initial conditions, and the parameters involved the update of the path parameters are as follows:

$$\begin{aligned}
&\eta_1(0) = \left[-5\ 0\ 0\right]^T,\ \eta_2(0) = \left[-5\ 5\ 0\right]^T,\ \eta_3(0) = \left[-5\ -5\ 0\right]^T,\\
&\omega_1(0) = \omega_2(0) = \omega_3(0) = \left[0\ 0\right]^T,\ \hat{\eta}_i(0) = \left[0\ 0\ 0\right]^T,\ \hat{X}_i(0) = \left[0\ 0\right]^T,\\
&P_{01i} = \mathrm{diag}(1,1,1),\ \ K_{01i} = \mathrm{diag}(1,1,1),\ \ P_{02i} = \mathrm{diag}(1,1),\\
&K_{02i} = (J_i(\eta_i)Q_i^{-1}(\eta_i))^T,\ s_i(0) = 2,\ k_{1i} = 1,\ k_{2i} = 0.5; k_{3i} = 1.5,\ k_{4i} = 2,\\
&k_{5i} = 2,\ \gamma_i = 1,\ \Gamma_i = \mathrm{diag}(1,1,1),\ \omega_0 = 0.2,\ \chi_1 = \chi_2 = 0,\ \chi_3 = 0.5,\ \varepsilon_s = 0.1.
\end{aligned} \tag{4.80}$$

The reference path of the center of the virtual structure is chosen as $\Gamma_0(s_0) = (s_0, 0)$. The distances from the place-holders to the center of the virtual structure are chosen as

$$\begin{aligned}
l_1(x_{d0}(s_1), y_{d0}(s_1)) &= (0,0),\\
l_2(x_{d0}(s_2), y_{d0}(s_2)) &= (3, 8 + 3\cos(0.5s_2)),\\
l_3(x_{d0}(s_3), y_{d0}(s_3)) &= (3, -8 + 3\sin(0.5s_3)).
\end{aligned} \tag{4.81}$$

These choices mean that the first place-holder coincides with the center of the virtual structure which moves on a straight line, and that the other two place-holders move on two sinusoidal paths. The robots' position and orientation, and path parameter errors are plotted in Figure 4.3. The path tracking errors and linear velocities are plotted in Figure 4.4. It is seen from these figures that each robot asymptotically track its own path generated by the virtual structure, and formation is successful, see plot of the path parameter errors. In addition, Figure 4.4 (bottom subplot) indicates that the first robot moves with a constant speed while the other two robots move with varying speeds to maintain the formation.

4.7 Notes and references

This chapter aims at a combination of the virtual structure and path-tracking approaches to derive a control system for formation control of a group of unicycle-type mobile robots. The conventional virtual structure approach is modified so that the formation shape can vary, i.e. the robots can change their relative positions with respect to the center of the virtual structure during the manoeuvre. The technique in Chapter 2, Section 2.2.2 was modified to design a global output feedback controller for each robot based on a global exponential/asymptotic observer. The output feedback controller is designed in such a way that the derivative of the path parameter is left as an additional control input to synchronize the formation motion. The control system is derived in four stages: first, the dynamics of the virtual structure are defined; second, the motion of the virtual structure is translated into the desired motion for each

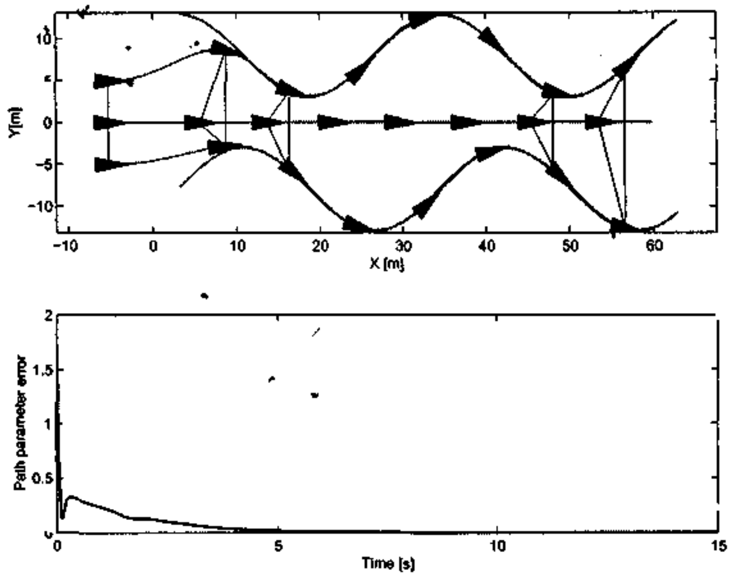
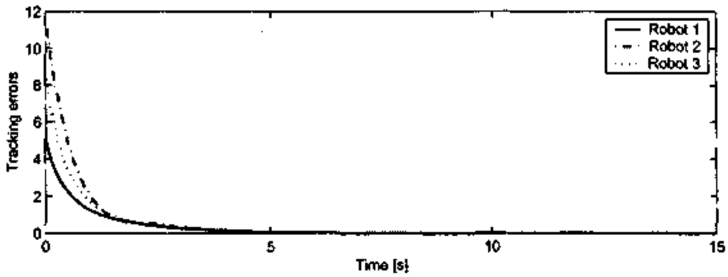


Fig. 4.3. Top: Robot position and orientation in (x,y) plane; Bottom: Path parameter errors in the form of $\sum_{1}^{3} \sqrt{(s_i - s_0)^2}$.

robot; third, output feedback tracking controller for each robot is derived; and finally, formation feedback is introduced from each robot to the virtual structure. The material in this chapter is based on the work in [51].



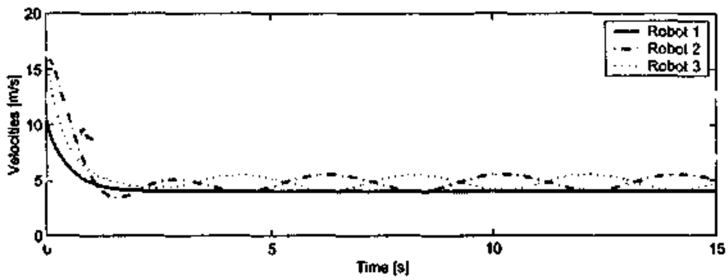


Fig. 4.4. Top: Tracking errors in the form of $\sqrt{x_{ei}^2 + y_{ei}^2 + \phi_{ei}^2}$; Bottom: Robot linear velocities v_i.

5

Bounded Formation Control of Multiple Agents with Limited Sensing

A constructive method, which is the base for the next two chapters, is presented to design bounded cooperative controllers that force a group of N mobile agents with limited sensing ranges to stabilize at a desired location, and guarantee no collisions between the agents in this chapter. The dynamics of each agent is described by a single integrator. The control development is based on new general potential functions, which attain the minimum value when the desired formation is achieved, and are equal to infinity when a collision occurs. A *p times differential bump function* given in Section A.7 is embedded into the potential functions to deal with the agent limited sensing ranges. An alternative to Barbalat's lemma given in Section A.6 is used to analyze stability of the closed loop system. Moreover, the controlled system exhibits multiple equilibrium points due to collision avoidance taken into account. We therefore investigate the behavior of equilibrium points by linearizing the closed loop system around those points, and show that critical points, other than the desired point for an agent, are unstable. The proposed formation stabilization solution is then extended to solve a formation tracking problem.

5.1 Problem statement

We consider a group of N mobile agents, of which each has the following dynamics

$$\dot{q}_i = u_i, i = 1, ..., N \tag{5.1}$$

where $q_i \in \mathbb{R}^n$ and $u_i \in D \subset \mathbb{R}^n$ are the state and control input of the agent i. We assume that $n > 1$ and $N > 1$. Here, we treat each agent as an autonomous point.

Control objective: Assume that at the initial time $t_0 \geq 0$ each agent starts at a different location, and that each agent has a different desired location, i.e. there exist strictly positive constants ϵ_1 and ϵ_2 such that for all $(i,j) \in \{1,2,...,N\}, i \neq j$

$$\begin{aligned} &\|q_i(t_0) - q_j(t_0)\| \geq \epsilon_1, \\ &\|q_{if} - q_{jf}\| \geq \epsilon_2 \end{aligned} \tag{5.2}$$

where $q_{if}, i = 1,..,N$, is the desired location of the agent i. Moreover, the agent i can only measure its own state and can only detect the other group members if these members are in a sphere, which is centered at the agent and has a radius of R_i larger than a strictly positive constant. Design the bounded control input u_i for each agent i such that each agent asymptotically approaches its desired location while avoids collisions with all other agents in the group, i.e. for all $(i,j) \in \{1,2,...,N\}, i \neq j, t \geq t_0 \geq 0$

$$\begin{aligned} &\|u_i(t)\| \leq \delta, \\ &\lim_{t\to\infty}(q_i(t) - q_{if}) = 0, \\ &\|q_i(t) - q_j(t)\| \geq \epsilon_3 \end{aligned} \tag{5.3}$$

where δ is a strictly positive constant and ϵ_3 is a positive constant.

5.2 Control design

Consider the following potential function

$$\varphi = \sum_{i=1}^{N}(\gamma_i + 0.5\beta_i) \tag{5.4}$$

where γ_i and β_i are the goal and related collision avoidance functions for the agent i specified as follows:
-The goal function is designed such that it puts penalty on the stabilization error for the agent, and is equal to zero when the agent is at its final position. A simple choice of this function is

$$\gamma_i = 0.5\|q_i - q_{if}\|^2 \tag{5.5}$$

-The related collision function β_i is chosen such that it is equal to infinity whenever any agents come in contact with the agent i , i.e. a collision occurs, and attains the minimum value when the agent i is at its desired location with respect to other group members belong to the set $\mathbb{N}_i$ agents, where $\mathbb{N}_i$ is the set that contains all the agents in the group except for the agent i. This function is chosen as follows:

$$\beta_i = \sum_{j \in \mathbb{N}_i} \beta_{ij} \tag{5.6}$$

where the function $\beta_{ij} = \beta_{ji}$ is a function of $\|q_{ij}\|^2/2$ and $\|q_{ijf}\|^2/2$, with $q_{ij} = q_i - q_j$ and $q_{ijf} = q_{if} - q_{jf}$, and enjoys the following properties:

$$\begin{aligned}
&1)\, \beta_{ij} = 0 \quad \text{if} \quad \|q_{ij}\| = \|q_{ijf}\|, \\
&2)\, \beta_{ij} > 0 \quad \text{if} \quad \|q_{ij}\| \neq \|q_{ijf}\| \quad \text{and} \quad 0 < \|q_{ij}\| < b_{ij}, \\
&3)\, \beta_{ij}|_{\|q_{ij}\|=0} = \infty, \\
&4)\, \beta'_{ij}|_{\|q_{ij}\|=\|q_{ijf}\|} = 0, \beta''_{ij}|_{\|q_{ij}\|=\|q_{ijf}\|} \geq 0,\ \beta'_{ij}|_{\|q_{ij}\|=0} = \infty, \\
&5)\, \beta_{ij} \leq \mu_1,\ |\beta'_{ij}| \leq \mu_2,\ |\beta''_{ij} q_{ij}^T q_{ij}| \leq \mu_3,\ \forall\, \mu_4 \leq \|q_{ij}\| \leq \mu_5, \\
&6)\, \beta_{ij} = 0, \beta'_{ij} = 0, \beta''_{ij} = 0 \quad \text{if} \quad \|q_{ij}\| \geq b_{ij}, \\
&7)\, \beta_{ij} \text{ is } p \text{ times differentiable with respect to } q_{ij}
\end{aligned} \tag{5.7}$$

where b_{ij} is a strictly positive constant such that $b_{ij} \leq \min(R_i, R_j)$, $\beta'_{ij} = \frac{\partial \beta_{ij}}{\partial(\|q_{ij}\|^2/2)}$ and $\beta''_{ij} = \frac{\partial^2 \beta_{ij}}{\partial(\|q_{ij}\|^2/2)^2}$, and $\mu_l, l = 1, ..., 5$ are positive constants, and $p \geq 2$ is a positive integer.

Remark 5.1. Properties 1), 2) and 3) imply that the function β_i is positive definite, is equal to zero when all agents are at their desired locations, and is equal to infinity when a collision between any agents in the group occurs. Property 4) and the function γ_i given in (5.5) ensure that the function φ attains the (unique) minimum value of zero when all the agents are at their desired positions. Property 5) is used to prove stability of the closed loop system. Property 6) ensures that the collision avoidance between the agents i and j is only taken into account when they are in their sensing ranges. Property 7) ensures that β_{ij} is at least twice differentiable with respect to q_{ij}.

There are many functions that satisfy all properties of β_{ij} given in (5.7). An example is

$$\begin{aligned}
\beta_{ij} = &\left(\frac{\|q_{ij}\|^2/2}{(\|q_{ijf}\|^2/2)^2} + \frac{1}{(\|q_{ij}\|^2/2)} - \frac{2}{(\|q_{ijf}\|^2/2)} \right)^2 \times \\
&h_{ij}\left(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\right)
\end{aligned} \tag{5.8}$$

where $h_{ij}\left(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\right)$ is a p (with $p \geq 2$) times differentiable bump function defined in Definition A.28. The derivative of φ along the solutions of (5.1) satisfies

$$\dot{\varphi} = \sum_{i=1}^{N} \Omega_i^T u_i \tag{5.9}$$

where

$$\Omega_i = q_i - q_{if} + \sum_{j \in \mathbf{N}_i} \beta'_{ij} q_{ij}. \tag{5.10}$$

From (5.9), a bounded control u_i for the agent i is simply designed as follows:

$$u_i = -c\Psi(\Omega_i) \tag{5.11}$$

where c is a positive constant, and $\Psi(\Omega_i)$ denotes a vector of bounded functions of elements of Ω_i in the sense that $\Psi(\Omega_i) = \left[\psi(\Omega_i^1)\ \psi(\Omega_i^2), ..., \psi(\Omega_i^l), ..., \psi(\Omega_i^n)\right]^T$ with Ω_i^l the l^{th} element of Ω_i, i.e. $\Omega_i = [\Omega_i^1\ \Omega_i^2 ... \Omega_i^l ... \Omega_i^n]^T$. The function $\psi(x)$ is a scalar, differentiable and bounded function, and satisfies

$$\begin{array}{ll} 1) & |\psi(x)| \leq M_1, \\ 2) & \psi(x) = 0 \quad \text{if } x = 0, \quad x\psi(x) > 0 \text{ if } x \neq 0, \\ 3) & \psi(-x) = -\psi(x), (x-y)[\psi(x) - \psi(y)] \geq 0, \\ 4) & |\frac{\psi(x)}{x}| \leq M_2, |\frac{\partial \psi(x)}{\partial x}| \leq M_3, \frac{\partial \psi(x)}{\partial x}|_{x=0} = 1 \end{array} \tag{5.12}$$

for all $x \in \mathbf{R}, y \in \mathbb{R}$, where M_1, M_2, M_3 are strictly positive constants. Some functions that satisfy the above properties are $\arctan(x)$ and $\tanh(x)$. Indeed, the control u_i is bounded, i.e. $\|u_i(t)\| \leq c\sqrt{n}M_1 := \delta, \forall t \geq t_0 \geq 0$.

Remark 5.2. When Ω_i defined in (5.10) is substituted into (5.11), the l^{th} element of the control u_i can be written as $u_i^l = c\psi\big(-(q_i^l - q_{if}^l) - \sum_{j\in \mathbf{N}_i} \beta'_{ij} q_{ij}^l\big)$ with q_i^l, q_{if}^l and q_{ij}^l being the l^{th} elements of q_i, q_{if}, and q_{ij}. The argument of ψ consists of two parts: $-(q_i^l - q_{if}^l)$ and $-\sum_{j\in \mathbf{N}_i} \beta'_{ij} q_{ij}^l$. The first part, $-(q_i^l - q_{if}^l)$, referred to as the attractive force plays the role of forcing the agent to its desired location. The second part, $-\sum_{j\in \mathbf{N}_i} \beta'_{ij} q_{ij}^l$, referred to as the repulsive force, takes care of collision avoidance for the agent i with the other agents. Moreover, the control u_i of the agent i given in (5.11) depends on only its own state, and the states of other neighbor agents j if these agents are in a sphere, which is centered at the agent and has a radius no greater than R_i because outside this sphere $\beta'_{ij} = 0$.

Now substituting (5.11) into (5.9) results in

$$\dot{\varphi} = -c\sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i). \tag{5.13}$$

Substituting (5.11) into (5.1) results in the closed loop system

$$\dot{q}_i = -c\Psi(\Omega_i), i = 1, ..., N. \tag{5.14}$$

We now state the main results of this chapter in the following theorem.

Theorem 5.3. *Assume that at the initial time $t_0 \geq 0$ each agent starts at a different location, and that each agent has a different desired location, i.e. the conditions given in (5.2) hold, the bounded controls given in (5.11) guarantee that no collisions between any agents can occur, the solutions of the closed loop system (5.14) exist and the agents asymptotically approach their desired positions (a set of equilibria) defined by $q_{if}, i = 1, ..., N$. Asymptotic convergence to the set of equilibria is "generic" (for generic initial conditions) and not global as one may erroneously think.*

5.3 Proof of Theorem 5.3

To prove Theorem 5.3, we first prove that no collisions between the agents can occur and that the agents asymptotically approach their target points or some critical points. Next, to investigate stability of the closed loop system (5.14) at these points, we linearize the closed loop system at these points. We then prove that only desired target points are unique asymptotic stable and that other critical points are unstable.

+Proof of no collisions and existence of solutions. From (5.13) and properties of the function ψ , see (5.12), we have $\dot{\varphi} \leq 0$, which implies that $\varphi(t) \leq \varphi(t_0), \forall t \geq t_0$. With definition of the function φ in (5.4) and its components in (5.5) and (5.6), we have

$$\sum_{i=1}^{N} \Big[\gamma_i(t) + 0.5 \sum_{j \in \mathbb{N}_i} \beta_{ij}(t)\Big] \leq \sum_{i=1}^{N} \Big[\gamma_i(t_0) + 0.5 \sum_{j \in \mathbb{N}_i} \beta_{ij}(t_0)\Big] \qquad (5.15)$$

for all $t \geq t_0 \geq 0$. From the first condition in (5.2) and Property 5) of β_{ij}, we have $\sum_{j \in \mathbb{N}_i} \beta_{ij}(t_0)$ is smaller than a strictly positive constant. Therefore the right hand side of (5.15) is bounded by a positive constant depending on the initial conditions. Boundedness of the right hand side of (5.15) implies that the left hand side of (5.15) must be also bounded. As a result, $\beta_{ij}(t)$ must be smaller than some positive constant depending on the initial conditions for all $t \geq t_0 \geq 0$. From properties of β_{ij}, see (5.7), $\|q_{ij}(t)\|$ must be larger than some positive constant depending on the initial conditions denoted by ϵ_3, i.e. there are no collisions for all $t \geq t_0 \geq 0$. Boundedness of the left hand side of (5.15) also implies that of $\|q_{ij}(t)\|$ and $\|q_i(t)\|$ for all $t \geq t_0 \geq 0$, i.e. the solutions of the closed loop system (5.14) exist.

+Equilibrium points. We will use Lemma A.24 to find the equilibrium points, which the trajectories of the closed loop system (5.14) converge to. Integrating both sides of (5.13) yields

$$\int_0^\infty \phi(t)dt = \varphi(0) - \varphi(\infty) \leq \varphi(0) \tag{5.16}$$

where $\phi(t) := c\sum_{i=1}^N \Omega_i(t)^T \Psi(\Omega_i(t)) = c\sum_{i=1}^N \sum_{l=1}^n \Omega_i^l(t)\psi(\Omega_i^l(t))$ with $\Omega_i^l(t)$ the l^{th} element of $\Omega_i(t)$. Indeed, the function $\phi(t)$ is scalar, nonnegative and differentiable. Now differentiating $\phi(t)$ along the solutions of the closed loop system (5.14) gives

$$\begin{aligned}\frac{d\phi(t)}{dt} = c\sum_{i=1}^N \sum_{l=1}^n \Big(\frac{\psi(\Omega_i^l)}{\Omega_i^l} + \frac{\partial\psi(\Omega_i^l)}{\partial\Omega_i^l}\Big)\Omega_i^l \Big\{ -\psi(\Omega_i^l) + \sum_{j\in\mathbb{N}_i}\Big[\beta'_{ij} \\ \times(-\psi(\Omega_i^l) + \psi(\Omega_j^l)) + \beta''_{ij}q_{ij}^l \sum_{h=1}^n(-\psi(\Omega_i^h) + \psi(\Omega_j^h))\Big]\Big\}\end{aligned} \tag{5.17}$$

where q_{ij}^l is the l^{th} element of q_{ij}. Since we have already proved that $\|q_{ij}(t)\|$ is bounded and is greater than a positive constant for all $t \geq t_0 \geq 0$, using properties of ψ and β_{ij} in (5.12) and (5.7) we have: 1) $|\psi(\Omega_i^l)/\Omega_i^l + \partial\psi(\Omega_i^l)/\partial\Omega_i^l| \leq \sigma_1$, 2) $|\beta'_{ij}| \leq \sigma_2$, 3) $|\beta''_{ij}q_{ij}^T q_{ij}| \leq \sigma_3$, and 4) $\Omega_i^l\psi(\Omega_i^l) + \Omega_j^l\psi(\Omega_j^l) \geq \Omega_j^l\psi(\Omega_i^l) + \Omega_i^l\psi(\Omega_j^l)$, with $\sigma_1, \sigma_2, \sigma_3$ positive constants. Applying these inequalities to (5.17) results in

$$\left|\frac{d\phi(t)}{dt}\right| \leq M_4 c\sum_{i=1}^N \sum_{l=1}^n \Omega_i^l\psi(\Omega_i^l) = M_4\phi(t) \tag{5.18}$$

where $M_4 = \sigma_1(1 + \sigma_2(N-1) + 2(\sigma_2 + \sigma_3 n))$. It is clear from (5.18) and (5.16) that the function $\phi(t)$ satisfies the conditions of Lemma A.25. Hence $\lim_{t\to\infty}\phi(t) = 0$, which implies from the definition of ϕ that

$$\lim_{t\to\infty} \Omega_i^T(t)\Psi(\Omega_i(t)) = 0, \forall i = 1, 2, ..., N. \tag{5.19}$$

Thanks to Property 2) of the function ψ, see (5.12), the limit equation (5.19) implies that

$$\lim_{t\to\infty} \Omega_i(t) = \lim_{t\to\infty}\Big[q_i(t) - q_{if} + \sum_{j\in\mathbb{N}_i}\beta'_{ij}(t)q_{ij}(t)\Big] = 0 \tag{5.20}$$

for all $i = 1, 2, ..., N$. The limit equation (5.20) implies that the state $q(t) = [q_1^T(t)\ q_2^T(t), ..., q_N^T(t)]^T$ converges to the manifold $\mathbb{M}$ of (5.14) contained in $\mathbf{E} = \{q \in \mathbb{R}^{n\times N}|_{\Omega=0}\}$ where $\Omega = [\Omega_1^T\ \Omega_2^T,\ ...,\ \Omega_N^T]^T$, i.e. on the surface where $\dot{\varphi} = 0$. This surface is continuous because we have already proved that $\|q_{ij}\| > 0, \forall(i,j) \in \{1, 2, ..., N\}, i \neq j$, i.e. β'_{ij} is continuous, see Properties of β_{ij} in (5.7). As the time t goes to infinity, it can be verified that one solution of (5.20) is $q_f = [q_{1f}^T\ q_{2f}^T, ..., q_{Nf}^T]^T$ since $\beta'_{ij}|_{\|q_{ij}\|=\|q_{ijf}\|} = 0$ (Property 4) of β_{ij}), and other solutions are denoted by $q_c = [q_{1c}^T\ q_{2c}^T, ..., q_{Nc}^T]^T$. It is noted that some elements of q_c can be equal to that of q_f. However, for simplicity we abuse the notation, i.e. we still denote that vector as q_c. Indeed, the vector q_c is such that

$$\Omega_i|_{q=q_c} = \Big[q_i - q_{if} + \sum_{j\in\mathbb{N}_i} \beta'_{ij} q_{ij}\Big]\Big|_{q=q_c} = 0 \tag{5.21}$$

for all $i = 1, ..., N$. Next, we will show that q_f is stable and q_c is unstable, by linearizing (5.14) at these points.

+Properties of equilibrium points. The closed loop system (5.14) can be written in a vector form as $\dot{q} = -c\Psi_q(q, q_f)$, and $\Psi_q(q, q_f) = [\Psi^T(\Omega_1), ..., \Psi^T(\Omega_i), ..., \Psi^T(\Omega_N)]^T$. Therefore, near an equilibrium point q_o, which can be either q_f or q_c, we have

$$\dot{q} = -c\, \partial\Psi_q(q, q_f)/\partial q\big|_{q=q_o}(q - q_o) \tag{5.22}$$

where

$$\frac{\partial\Psi_q(q, q_f)}{\partial q} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \cdots & \cdots & \Delta_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{i1} & \cdots & \Delta_{ii} & \cdots & \Delta_{iN} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \cdots & \cdots & \cdots & \Delta_{NN} \end{bmatrix} \tag{5.23}$$

with $\Delta_{ij} = \frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\frac{\partial\Omega_i}{\partial q_i}$, $(i, j) \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all agents in the group. A simple calculation shows that for all $i = 1, ..., N, j \in \mathbb{N}_i, j \neq i$

$$\frac{\partial\Omega_i}{\partial q_i} = \Big(1 + \sum_{i\in\mathbb{N}_i} \beta'_{ij}\Big) I_n + \sum_{j\in\mathbb{N}_i} \beta''_{ij} q_{ij} q_{ij}^T,$$
$$\frac{\partial\Omega_i}{\partial q_j} = -\beta'_{ij} I_{n\times n} - \beta''_{ij} q_{ij} q_{ij}^T. \tag{5.24}$$

Let $\mathbb{N}^*$ be the set of the agents such that if the agents i and j belong to the set $\mathbb{N}^*$ then $\|q_{ij}\| < b_{ij}$. Next we will show that q_f is asymptotically stable and that q_c is unstable.

- Proof of q_f being asymptotic stable. Using properties of β_{ij} and ψ in (5.7) and (5.12), we have from (5.24) that for all $i = 1, ..., N, i \neq j$:

$$\frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\bigg|_{q=q_f} = I_n, \frac{\partial\Omega_i}{\partial q_i}\bigg|_{q=q_f} = I_n + \sum_{j\in\mathbb{N}_i} \beta''_{ijf} q_{ijf} q_{ijf}^T,$$
$$\beta'_{ijf} = 0, \frac{\partial\Omega_i}{\partial q_j}\bigg|_{q=q_f} = -\beta''_{ijf} q_{ijf} q_{ijf}^T \tag{5.25}$$

where $\beta'_{ijf} = \beta'_{ij}\big|_{q_{ij}=q_{ijf}}$ and $\beta''_{ijf} = \beta''_{ij}\big|_{q_{ij}=q_{ijf}}$. We consider the Lyapunov function candidate $V_f = 0.5\|q - q_f\|^2$ whose derivative along the solutions of the linearized closed loop system (5.22) with q_o replaced by q_f, and using (5.25) satisfies $\dot{V}_f = -c\sum_{i=1}^{N}\|q_i - q_{if}\|^2 - c\sum_{(i,j)\in \mathbb{N}, i\neq j}\beta''_{ijf}\big(q^T_{ijf}(q_{ij} - q_{ijf})\big)^2$. Since $\beta''_{ijf} \geq 0$, see Property 4) in (5.7), we have $\dot{V}_f \leq -c\sum_{i=1}^{N}\|q_i - q_{if}\|^2 = -2cV_f$, which implies that q_f is asymptotically stable.

- *Proof of q_c being unstable.* Again using properties of β_{ij} and ψ in (5.7) and (5.12), we have from (5.24) that for all $i = 1, ..., N, i \neq j$:

$$\frac{\partial \Psi(\Omega_i)}{\partial \Omega_i}\bigg|_{q=q_c} = I_n,\ \frac{\partial \Omega_i}{\partial q_i}\bigg|_{q=q_c} = \big(1 + \sum_{j\in \mathbb{N}_i}\beta'_{ijc}\big)I_n + \sum_{j\in \mathbb{N}_i}\beta''_{ijc}q_{ijc}q^T_{ijc},\ \frac{\partial \Omega_i}{\partial q_j}\bigg|_{q=q_c} = -\beta'_{ijc} - \beta''_{ijc}q_{ijc}q^T_{ijc} \tag{5.26}$$

where $q_{ijc} = q_{ic} - q_{jc}$, $\beta'_{ijc} = \beta'_{ij}\big|_{q_{ij}=q_{ijc}}$ and $\beta''_{ijc} = \beta''_{ij}\big|_{q_{ij}=q_{ijc}}$. Since the related collision avoidance functions β_i, see (5.6), are specified in terms of relative distances between agents and it is extremely difficult to obtain q_c explicitly by solving (5.21), it is very difficult to use the Lyapunov function candidate $V_c = 0.5\|q - q_c\|$ to investigate stability of (5.22) at q_c. Therefore, we consider the following Lyapunov function candidate

$$\bar{V}_c = 0.5\|\bar{q} - \bar{q}_c\|^2 \tag{5.27}$$

where $\bar{q} = [q^T_{12}, q^T_{13}, ...q^T_{1N}, q^T_{23}, ..., q^T_{2N}, ..., q^T_{N-1,N}]^T$ and $\bar{q}_c = [q^T_{12c}, q^T_{13c}, ...q^T_{1Nc}, q^T_{23c}, ..., q^T_{2Nc}, ..., q^T_{N-1,Nc}]^T$. Differentiating both sides of (5.27) along the solution of the linearized closed loop system (5.22) with q_o replaced by q_c gives

$$\dot{\bar{V}}_c = -c\sum_{(i,j)\in \mathbb{N}\backslash \mathbb{N}^*}\|q_{ij} - q_{ijc}\|^2 - c\sum_{(i,j)\in \mathbb{N}^*}(1 + N\beta'_{ijc})\times \|q_{ij} - q_{ijc}\|^2 - cN\sum_{(i,j)\in \mathbb{N}^*}\beta''_{ijc}\big(q^T_{ijc}(q_{ij} - q_{ijc})\big)^2 \tag{5.28}$$

where $i \neq j$ and (5.26) has been used. To investigate stability properties of $\bar{q}_c$ based on (5.28), we will use (5.21). Define $\Omega_{ijc} = \Omega_{ic} - \Omega_{jc}$, $\forall(i,j) \in \{1, ..., N\}$, $i \neq j$ where $\Omega_{ic} = \Omega_i|_{q=q_c} = 0$, see (5.21). Therefore $\Omega_{ijc} = 0$. Hence $\sum_{(i,j)\in \mathbb{N}^*} q^T_{ijc}\Omega_{ijc} = 0, i \neq j$, which by using (5.21) is expanded to

$$\sum_{(i,j)\in \mathbb{N}^*}\big(q^T_{ijc}(q_{ijc} - q_{ijf}) + N\beta'_{ijc}q^T_{ijc}q_{ijc}\big) = 0$$
$$\Rightarrow \sum_{(i,j)\in \mathbb{N}^*}(1 + N\beta'_{ijc})q^T_{ijc}q_{ijc} = \sum_{(i,j)\in \mathbb{N}^*}q^T_{ijc}q_{ijf} \tag{5.29}$$

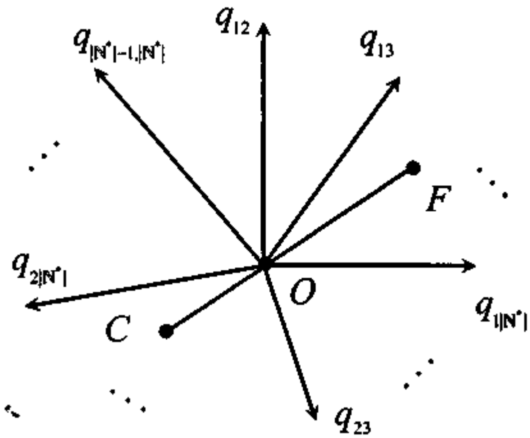


Fig. 5.1. *Illustration of equilibrium points.*

where $i \neq j$. The sum $\sum_{(i,j)\in\mathbb{N}^*} q_{ijc}^T q_{ijf}$ is strictly negative since at the point where $q_{ij} = q_{ijf}$, $\forall(i,j) \in \mathbb{N}^*, i \neq j$ (the point F in Fig. 5.1) all attractive and repulsive forces are equal to zero while at the point where $q_{ij} = q_{ijc}$ $\forall(i,j) \in \mathbb{N}^*, i \neq j$ (the point C in Fig. 5.1) the sum of attractive and repulsive forces are equal to zero (but attractive and repulsive forces are nonzero). Therefore the point where $q_{ij} = 0$, $\forall(i,j) \in \mathbb{N}^*, i \neq j$ (the point O in Fig. 5.1) must locate between the points F and C for all $(i,j) \in \mathbb{N}^*, i \neq j$. That is there exists a strictly positive constant b such that $\sum_{(i,j)\in\mathbb{N}^*} q_{ijc}^T q_{ijf} < -b$, which is substituted into (5.29) to yield

$$\sum_{(i,j)\in\mathbb{N}^*} (1 + N\beta'_{ijc}) q_{ijc}^T q_{ijc} < -b, i \neq j. \tag{5.30}$$

Since $q_{ijc}^T q_{ijc} > 0, \forall(i,j) \in \mathbb{N}^*, i \neq j$, there exists a nonempty set $\mathbb{N}^{**} \subset \mathbb{N}^*$ such that for all $(i,j) \in \mathbb{N}^{**}, i \neq j$, $(1 + N\beta'_{ijc})$ is strictly negative, i.e. there exists a strictly positive constant b^{**} such that $(1 + N\beta'_{ijc}) < -b^{**}$, $\forall(i,j) \in \mathbb{N}^{**}, i \neq j$. We now write (5.28) as

$$\begin{aligned} \dot{V}_c = -c\Bigg[\sum_{(i,j)\in\mathbb{N}\setminus\mathbb{N}^*} \|q_{ij} - q_{ijc}\|^2 + \sum_{(i,j)\in\mathbb{N}^*\setminus\mathbb{N}^{**}} (1 + N\beta'_{ijc}) \times \\ \|q_{ij} - q_{ijc}\|^2 + N \sum_{(i,j)\in\mathbb{N}^*} \beta''_{ijc}\big(q_{ijc}^T(q_{ij} - q_{ijc})\big)^2 \Bigg] - \\ c \sum_{(i,j)\in\mathbb{N}^{**}} (1 + N\beta'_{ijc}) \|q_{ij} - q_{ijc}\|^2 \end{aligned} \tag{5.31}$$

where $i \neq j$. We now define a subspace such that $q_{ij} - q_{ijc} = 0,\ \forall (i,j) \in \mathbb{N} \setminus \mathbb{N}^{**}$ and $q_{ijc}^T(q_{ij} - q_{ijc}) = 0,\ \forall (i,j) \in \mathbb{N}^*, i \neq j$. In this subspace, we have

$$\bar{V}_c = 0.5 \sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2,$$

$$\dot{\bar{V}}_c = -c \sum_{(i,j)\in \mathbb{N}^{**}} (1 + N\beta'_{ijc})\|q_{ij} - q_{ijc}\|^2 \geq 2cb^{**}\bar{V}_c \tag{5.32}$$

where we have used $(1 + N\beta'_{ijc}) < -b^{**},\ \forall (i,j) \in \mathbb{N}^{**}, i \neq j$. Clearly (5.32) implies that

$$\sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| \geq \sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| e^{cb^{**}(t-t_0)} \tag{5.33}$$

for all $i \neq j,\ t \geq t_0 \geq 0$. Now assume that q_c is a stable equilibrium point of the closed loop system (5.14), i.e. $\lim_{t\to\infty} \|q_i(t) - q_{ic}\| = d_i, \forall i \in \mathbb{N}$ with d_i a nonnegative constant. Note that $\mathbb{N}^{**} \subset \mathbb{N}$, we have $\lim_{t\to\infty} \|q_i(t) - q_{ic}\| = d_i, \forall i \in \mathbb{N}^{**}$, which implies that $\lim_{t\to\infty} \sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}, \forall (i,j) \in \mathbb{N}^{**}, i \neq j$ with d^{**} a nonnegative constant, since $q_{ij} = q_i - q_j$ and $q_{ijc} = q_{ic} - q_{jc}$. This contradicts (5.33) for the case $\sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| \neq 0$, since the right hand side of (5.33) is divergent (so does the left hand side). For the case $\sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| = 0$, there would be no contradiction. However this case is never observed in practice since the ever-present physical noise would cause $\|q_{ij}(t^*) - q_{ijc}\|$ for some $(i,j) \in \mathbb{N}^{**}, i \neq j$ to be different from 0 at the time $t^* \geq t_0$. We now write (5.33) as

$$\sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| \geq \sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\| e^{cb^{**}(t-t^*)} \tag{5.34}$$

for all $i \neq j,\ t \geq t^* \geq t_0 \geq 0$. Since $\sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\| \neq 0$, the right hand side of (5.34) is divergent (so does the left hand side). This contradicts $\lim_{t\to\infty} \sum_{(i,j)\in \mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}, \forall (i,j) \in \mathbb{N}^{**}, i \neq j$. Therefore q_c must be an unstable equilibrium point of the closed loop system (5.14). Proof of Theorem 5.3 is completed.

5.4 Simulations

We carry out a simulation with $n = 2, N = 30$. The agents are initialized randomly in a circle, which is centered at the origin and has a radius of 1. The desired formation is specified in shape, location and orientation as $q_{if} = R_f[\sin((i-1)2\pi/N); \cos((i-1)2\pi/N)]$, $i = 1, ..., N$ with $R_f = 10$, i.e. the desired formation is a polygon whose vertices are uniformly distributed on a circle, which is centered at the origin and has a radius of 10. All agents have the same sensing range: $R_i = 2$. The function β_{ij} is chosen as in (5.8). The parameters of the p times differential bump functions are $p = 2, a_{ij} = 1, b_{ij} = 1.5$. The function ψ is taken as arctan. The control gain is chosen as $c = 2$. Simulation results are plotted in Fig. 5.2. It is seen that all agents nicely approach their desired locations. Since the agents initialize pretty close to each other, they quickly move away from each other then approach their desired locations, see the top-left figure in Fig. 5.2, where the trajectory of the agent 1 is plotted in the thick line, and agents 1 and 2 are indicated by 1 and 2. For clarity, only the control input $u_1 = [u_{1x}\ u_{1y}]^T$ is plotted in the top-right figure of Fig. 5.2. Noticing that the values of the continuous controls u_{1x} and u_{1y} are in the range $\pm\pi$. The bottom-right figure in Fig. 5.2 plots a 'mean-product' distance, $\text{dist}_{\text{all}} = \left(\prod_{(i,j)\in\mathbb{N}} \|q_{ij}\|\right)^{N(N-1)/2}$, see the thick line, and the distances between the agent 1 and other agents in the group. Clearly, no collisions between any agents occurred since dist_{all} is larger than zero for all simulation time. The bottom-left figure in Fig. 5.2 plots the functions $\beta_{1j}, j = 2, ..., N$. It is noted that all β_{1j} vanish when $\|q_{1j}\|$ are larger or equal to 1.5.

5.5 Extension to formation tracking

This section extends the results developed in the previous sections to solve the problem of designing a control input u_i for each agent i that forces the group of N mobile agents whose dynamics are given in (5.1) to track a moving Desired Formation Graph (DFG), in the sense that the DFG is allowed to move on a common desired trajectory $q_{od}(s)$ with s being the common reference trajectory parameter defined in the fixed coordinate system Π_F, see Fig. 5.3 and Fig. 5.4. We consider the DFG whose center moves along the common reference trajectory $q_{od}(s)$. We assume that $q_{od}(s)$ is regular in the sense that it is single valued and its first derivative exists and is bounded. Since the DFG under consideration is only representative, the center does not have to be the "true" center of the DFG but can be any convenient point. When the DFG moves along the trajectory $q_{od}(s)$, the vertex i of DFG generates the reference trajectory $q_{id}(s)$ for the agent i to track. We limit our consideration to two- and three-dimensional (2D and 3D) spaces, which are most common in practice. The control objective is now stated as follows.

Control objective. Assume that at the initial time $t_0 \geq 0$, for all $(i, j) \in \{1, 2, ...N\}$, $i \neq j$, $s \in \mathbb{R}$ there exist strictly positive constants ε_1, ε_2 and ε_3 such that

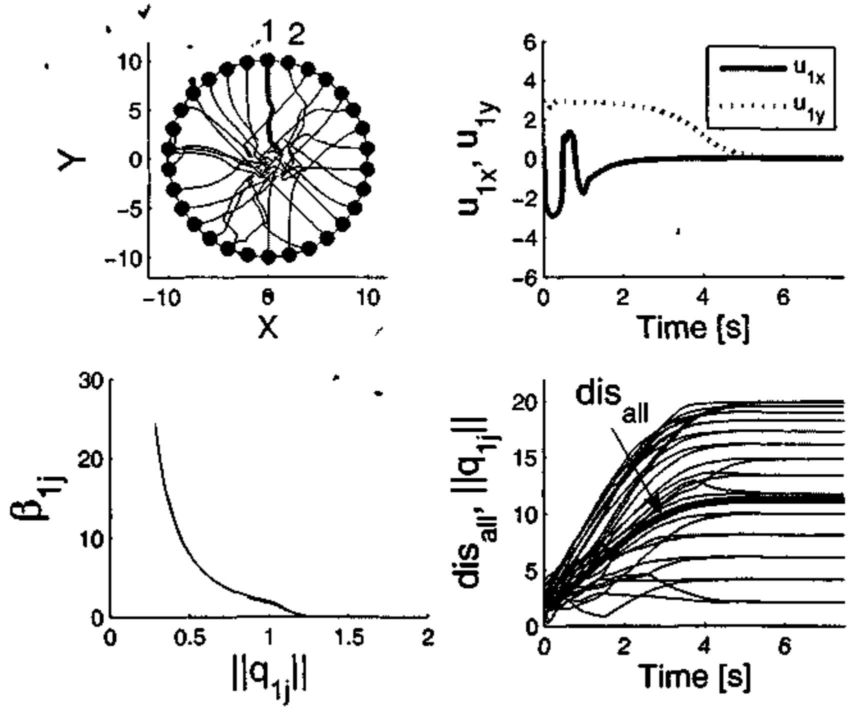


Fig. 5.2. *Simulation results.*

$$\|q_i(t_0) - q_j(t_0)\| \geq \varepsilon_1, \ \|q_{id}(s) - q_{jd}(s)\| \geq \varepsilon_2,$$
$$\|\frac{\partial q_{id}(s)}{\partial s}\| \leq \varepsilon_3 \tag{5.35}$$

Moreover, the agent i can only measure its own state and can only detect the other group members if these members are in a sphere with a radius of R_i larger than a strictly positive constant, and centered at the agent i. Design the control input u_i for each agent i such that

$$\lim_{t\to\infty}(q_i(t) - q_{id}(s)) = 0, \quad \|q_i(t) - q_j(t)\| \geq \varepsilon_4 \tag{5.36}$$

for all $(i, j) \in \{1, 2, ...N\}, i \neq j, s \in \mathbb{R}$, where ε_4 is a positive constant.

Control design. Let us construct $q_{id}(s)$. Let the moving coordinate system Π_M, of which the origin $\widehat{O}$ coincides with the center of the desired formation graph, move along $q_{od}(s)$. Let $\widehat{q}_{id}$ be the coordinate vector of the vertex i of the DFG in the moving coordinate system. We then have

$$\widehat{q}_{id} = J(\bullet)(q_{id}(s) - q_{od}(s)) \tag{5.37}$$

where $J(\bullet)$ whose elements depend on $\partial q_{od}(s)/\partial s$ and is the rotational (invertible) matrix, which describes the rotation of Π_M with respect to Π_F, and is such that $\delta_1 \leq \|J(\bullet)\| \leq \delta_2$, $\delta_3 \leq \|J(\bullet)^{-1}\| \leq \delta_4$ with $\delta_i, i = 1, .., 4$ strictly positive constants. Therefore, by specifying $\widehat{q}_{id}$ in Π_M, $q_{id}(s)$ in Π_F for the agent i can be calculated from (5.37). Similarly, in Π_M the coordinate vector of each agent i satisfies

$$\widehat{q}_i = J(\bullet)(q_i - q_{od}(s)). \tag{5.38}$$

From (5.37) and (5.38), we can see that the first two conditions in (5.35) imply the following condition

$$\|\widehat{q}_i(t_0) - \widehat{q}_j(t_0)\| \geq \widehat{\varepsilon}_1, \; \|\widehat{q}_{id}(s) - \widehat{q}_{jd}(s)\| \geq \widehat{\varepsilon}_2 \tag{5.39}$$

where $\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_2$ are some strictly positive constants. Moreover, the tracking control goal specified in (5.36) is achieved by designing the control u_i for each agent i such that

$$\lim_{t\to\infty}(\widehat{q}_i(t) - \widehat{q}_{id}(s)) = 0, \quad \|\widehat{q}_i(t) - \widehat{q}_j(t)\| \geq \widehat{\varepsilon}_4 \tag{5.40}$$

where $\widehat{\varepsilon}_4$ is a positive constant, and by letting the DFG move along the common reference trajectory via giving $\dot{s}$ some desired value.

Now differentiating both sides of (5.38) gives

$$\dot{\widehat{q}}_i = \widehat{u}_i \tag{5.41}$$

where $\widehat{u}_i$ is the new control, and we have chosen the control u_i as

$$u_i = \dot{q}_{od}(s) + J(\bullet)^{-1}(\widehat{u}_i - \dot{J}(\bullet)(q_i - q_{od}(s))). \tag{5.42}$$

The problem of designing $\widehat{u}_i$ for (5.41) to achieve (5.40) under (5.39) is exactly the same as the control objective in Section 5.1. Therefore, the control design in Section 5.2 can be used directly to design a bounded control $\widehat{u}_i$ to achieve the goal (5.40). After $\widehat{u}_i$ is designed, the actual tracking control u_i is calculated from (5.42). Let us give the expression of the rotational matrix $J(\bullet)$ in 2D and 3D spaces.

Two-dimensional space. Consider the moving coordinate frame, $\widehat{O}\widehat{X}\widehat{Y}$ attached to the DFG, as shown in Fig. 5.3. The origin $\widehat{O}$ coincides with the center of the graph, and is on the common reference trajectory $q_{od}(s) = [x_{od}(s) \; y_{od}(s)]^T$. The $\widehat{O}\widehat{X}$ and $\widehat{O}\widehat{Y}$ axes are tangential and perpendicular to

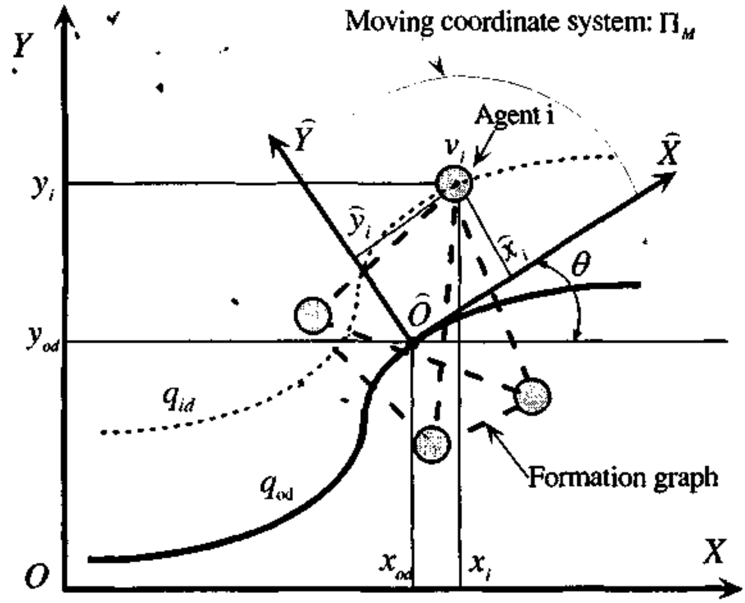


Fig. 5.3. *Formation coordinates in 2D.*

the reference trajectory $q_{od}(s)$. Therefore the angle θ between $\widehat{O}\widehat{X}$ and OX is calculated as $\theta = \arctan(y_d^{'}/x_d^{'})$, where $\bullet' \triangleq \partial \bullet /\partial s$. Hence the rotational matrix $J(\bullet)$ is given by $J(\bullet) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$, which is indeed invertible for all $\theta \in \mathbb{R}$, and $\|J(\bullet)\| = 1$.

Three-dimensional space. Consider the moving coordinate frame, $\widehat{O}\widehat{X}\widehat{Y}\widehat{Z}$, attached to the DFG as shown in Fig. 5.4. The coordinate frame $\widehat{O}\,X_1Y_1Z_1$ is parallel to $OXYZ$. The origin $\widehat{O}$ coincides with the center of the graph, and is on the reference trajectory $q_{od}(s) = [x_{od}(s)\ y_{od}(s)\ z_{od}(s)]^T$. The $\widehat{O}\widehat{X}$, $\widehat{O}\widehat{Y}$ and $\widehat{O}\widehat{Z}$ axes coincide with the unit tangent vector $\mathbf{t}$, the unit principal vector $\mathbf{n}$, and the unit binormal vector $\mathbf{b}$ of the trajectory $q_{od}(s)$ at the point $\widehat{O}$. These unit vectors form a positively oriented triple of vectors called the moving triad, and are given by $\mathbf{t} = \mathbf{q}'_{od}/\|\mathbf{q}'_{od}\|$, $\mathbf{n} = \mathbf{t}'/\|\mathbf{t}'\|$, $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, where $\times$ stands for the vector cross product operation, $(\mathbf{i},\ \mathbf{j},\ \mathbf{k})$ are the unit vectors of the $OXYZ$ coordinate frame. Let $(\xi_{i1},\ \xi_{i2}, \xi_{i3}), i = 1, 2, 3$ be the directional cosines of $\widehat{O}\widehat{X}$, $\widehat{O}\widehat{Y}$ and $\widehat{O}\widehat{Z}$ with respect to the fixed axes OX, OY and OZ, respectively. This notation means that if we denote $(\theta_{i1},\ \theta_{i2}, \theta_{i3}), i = 1, 2, 3$ as the angles

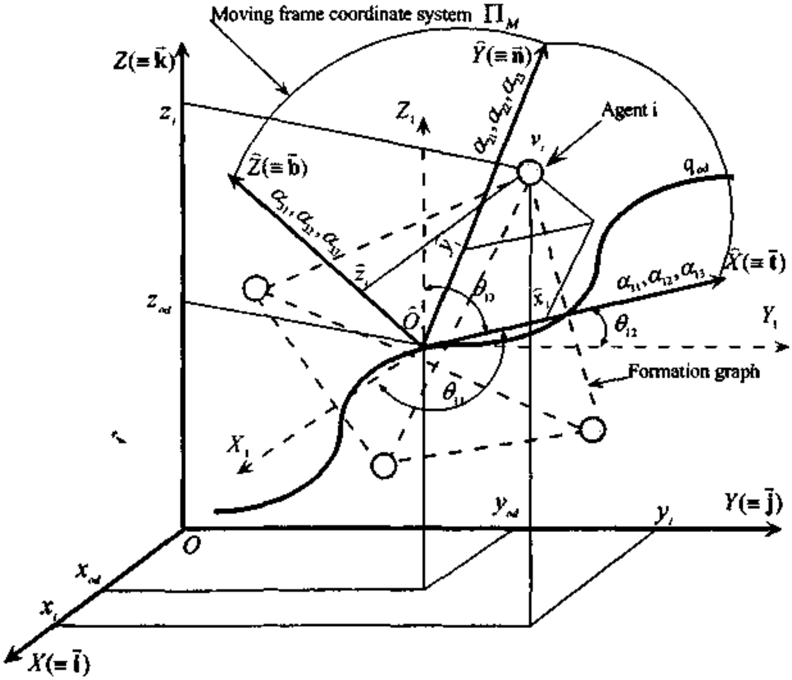


Fig. 5.4. *Formation coordinates in 3D.*

between the axes $\widehat{OX}$, $\widehat{OY}$ and $\widehat{OZ}$, and the axes OX, OY and OZ (see Fig. 5.4 for representation of θ_{11}, θ_{12} and θ_{13}), we have $\xi_{ij} = \cos(\theta_{ij})$, $\forall i, j \in \{1,2,3\}$. The rotational matrix $J(\bullet)$ is given by $J(\bullet) = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{bmatrix}$. It is shown in [52] that the determinant of $J(\bullet)$ is equal to 1, i.e. $J(\bullet)$ is globally invertible, and $\|J(\bullet)\| = 1$.

5.6 Notes and references

The main problem with the decentralized approach, when collision avoidance is taken into account, is that it is extremely difficult to predict and control the critical points of the controlled systems. An interesting work addressing geometric formation based on Voronoi partition optimization is given in [53] but the final arrangement of the agents cannot be foretold due to locality of Lloy's algorithm. Recently, a method based on a different navigation function from [32] provided a centralized formation stabilization control design strategy is proposed in [46]. This work is extended to a decentralized version in [47]. However, the potential function, which possesses all properties of a navigation function (see [32]), is finite (but its gradient with respect to the system states can be unbounded) when a collision occurs. This complicates analysis of collision avoidance. Moreover, the formation is stabilized to any point in workspace instead of being "tied" to a fixed coordinate frame. In [32], [46] and [47], the tuning constants, which are crucial to guarantee that the only desired equilibrium points are asymptotic stable and that the other critical points are unstable, are extremely difficult to obtain. Moreover, the control design methods (e.g. [36], [48], [37], [54]) based on the potential functions that are equal to infinity when a collision occurs exhibit very large control efforts if the agents are close to each other. The bounded formation controllers presented in this chapter require no specialities of tuning design constants. The material in this chapter is based on the work in [55], [56] and [57].

6

Formation Control of Mobile Robots with Limited Sensing: State Feedback

Based on the material presented in Chapter 5, a constructive method is presented to design cooperative controllers that force a group of N unicycle-type mobile robots with limited sensing ranges to perform desired formation tracking, and guarantee no collisions between the robots. Each robot requires only measurement of position and velocity of itself, and those of the robots within its sensing range for feedfack. Physical dimensions and dynamics of the robots are also considered in the control design. Smooth and p times differential bump functions given in Section A.7 are incorporated into novel potential functions to design a formation tracking control system. Despite the robot limited sensing ranges, no switchings are needed to solve the collision avoidance problem. Simulations illustrate the results.

6.1 Problem statement

6.1.1 Robot dynamics

We consider a group of N mobile robots, of which each has the following dynamics (see Chapter 1, Section 1.3.1):

$$\begin{aligned}
\dot{x}_i &= v_i \cos(\phi_i) \\
\dot{y}_i &= v_i \sin(\phi_i) \\
\dot{\phi}_i &= w_i \\
\overline{M}_i \dot{\varpi}_i &= -\overline{C}_i(w_i)\varpi_i - \overline{D}_i \varpi_i + \overline{B}_i \tau_i, i = 1, \ldots, N
\end{aligned} \tag{6.1}$$

where all the state variables and parameters are defined in Section 1.3.1, Chapter 1.

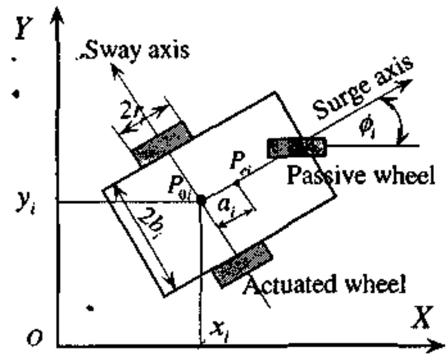


Fig. 6.1. Robot parameters.

6.1.2 Formation control objective

In this chapter, we will study a formation control problem under the following assumption:

Assumption 6.1 *1. The robot i, see Figure 6.1, has a physical safety circular area, which is centered at the point P_{0i} and has a radius $\underline{R}_i$, and has a circular communication area, which is centered at the point P_{0i} and has a radius $\overline{R}_i$, see Fig. 6.2. The radius $\overline{R}_i$ is strictly larger than $\underline{R}_i + \underline{R}_j$, $j = 1, ..., N, j \neq i$.*

2. *The robot i broadcasts its state, $(x_i, y_i, \phi_i, \varpi_i)$, and its reference trajectory, q_{id}, in its communication area. Moreover, the robot i can receive the states and reference trajectories broadcasted by other robots $j, j = 1, ..., N, j \neq i$ in the group if the points P_{0j} of these robots are in the communication area of the robot i.*
3. *The dimensional terms, (r_i, a_i, b_i), of the robot i are known to the robot i. The terms involved with mass, inertia and damping, $(m_{11i}, m_{12i}, d_{11i}, d_{22i}, c_i)$, of the robots are unknown but constant.*
4. *At the initial time $t_0 \geq 0$, each robot starts at a location that is outside of the safety areas of other robots in the group, i.e. there exists a strictly positive constant ε_1 such that*

$$\|q_i(t_0) - q_j(t_0)\| - (\underline{R}_i + \underline{R}_j) \geq \varepsilon_1, \ \forall(i, j) \in (1, 2..., N), \ i \neq j \tag{6.2}$$

where $q_i = [x_i \ y_i]^T$.

5. *The reference trajectory for the robot i is $q_{id} = [x_{id} \ y_{id}]^T$, which is generated by*

$$q_{id} = q_{od}(s_{od}) + l_i \tag{6.3}$$

where $q_{od}(s_{od}) = [x_{od}(s_{od})\ y_{od}(s_{od})]^T$ *is referred to as the common reference trajectory with* s_{od} *being the common trajectory parameter, and* l_i *is a constant vector. The trajectory* q_{od} *satisfies the following conditions*

$$\lim_{t\to\infty} u_{od}^2(t) \neq 0,\ u_{od} = \sqrt{x_{od}'^2 + y_{od}'^2}\,\dot{s}_{od},$$
$$\sqrt{x_{od}'^2 + y_{od}'^2} > 0,\ |u_{od}(t)| \leq u_{od}^{max} \tag{6.4}$$

where $x'_{od} = \frac{\partial x_{od}}{\partial s_{od}}$, $y'_{od} = \frac{\partial y_{od}}{\partial s_{od}}$, *and* u_{od}^{max} *is a strictly positive constant. Moreover,* $\dot{u}_{od}, \ddot{u}_{od}$ *are also bounded. The constant vectors* $l_i, i = 1, 2, ..., N$ *satisfy*

$$\|l_i - l_j\| - (\underline{R}_i + \underline{R}_j) \geq \epsilon_2,\ \forall (i,j) \in (1, 2..., N),\ i \neq j \tag{6.5}$$

where ϵ_2 *is a strictly positive constant.*

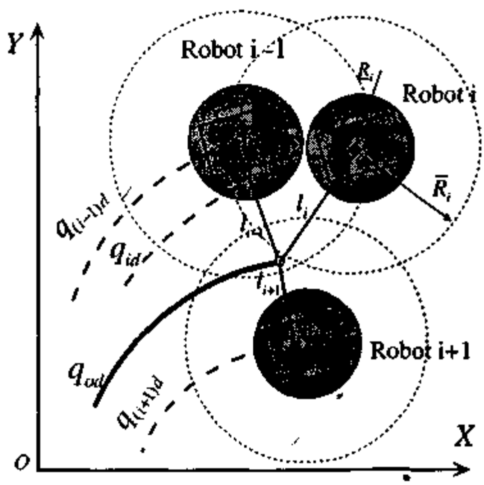


Fig. 6.2. Formation setup.

Remark 6.2. Items 1) and 2) in Assumption 6.1 specify the way each robot communicates with other robots in the group within its communication range. In Fig. 6.2, the robots i and $i-1$ are communicating with each other since the points $P_{0,i-1}$ and P_{0i} are in the communication areas of the robots i and $i-1$, respectively. The robots i and $i+1$ are not communicating with each other since the points P_{0i} and $P_{0,i+1}$ are not in the communication areas of the robots $i+1$ and i, respectively. Similarly, the robots $i-1$ and $i+1$ are not communicating with each other. Item 3) makes sense in practice because the dimensional terms can be easily predetermined while the terms involved mass, inertia and damping are often difficult to predetermine. Moreover, the assumptions in this item mean that the matrix $\overline{B}_i$ is known, while the matrices $\overline{M}_i, \overline{D}_i$ and coefficients of the entries of the matrix $\overline{C}_i(w_i)$ are unknown but constant. In item 5), the constant vectors $l_i, i = 1, ..., N$ specifies the desired formation configuration with respect to the earth-fixed frame OXY. The condition (6.4) implies that the common reference q_{od} is regular and its velocity u_{od}, which specifies how the desired formation, whose configuration is determined by l_i, moves along q_{od}, is bounded and satisfies a persistent excitation condition, i.e. the desired formation always moves along the common reference trajectory, q_{od}. The condition (6.5) specifies feasibility of the reference trajectories $q_{id}, i = 1, 2, ..., N$ (recall from (6.3) that $q_{id} - q_{jd} = l_i - l_j, \forall i \neq j$) due to physical safety circular areas of the robots. Finally, all the robots in the group require knowledge of the common reference trajectory q_{od} since this trajectory specifies how the whole formation should move with respect to the earth-fixed frame OXY.

Formation control objective: Under Assumption 6.1, design the control input τ_i and update laws for all terms involved mass, inertia and damping ($m_{11i}, m_{12i}, d_{11i}, d_{22i}$ and c_i) for each robot i such that each robot asymptotically tracks its desired reference trajectory q_{id} while avoids collisions with all other robots in the group, i.e. for all $(i, j) \in \{1, 2, ..., N\}, i \neq j, t \geq t_0 \geq 0$

$$\begin{aligned}
&\lim_{t\to\infty}(q_i(t) - q_{id}(t)) = 0, \\
&\lim_{t\to\infty}(\phi_i(t) - \phi_{id}(t)) = 0, \\
&\|q_i(t) - q_j(t)\| - (\underline{R}_i + \underline{R}_j) \geq \epsilon_3 \qquad (6.6)
\end{aligned}$$

where $\phi_{id}(t) = \arctan(y'_{od}/x'_{od})$, and ϵ_3 is a positive constant.

6.2 Control design

Since the robot dynamics (6.1) is of a strict feedback form [12] with respect to the robot linear velocity v_i and angular velocity w_i, we will use the backstepping technique [12] to design the control input τ_i. The control design is divided into two main stages. At the first stage, we consider the first three equations of (6.1) with v_i and w_i being considered as immediate controls. At the second stage, the actual control τ_i will be designed.

6.2.1 Stage I

Since the robot is underactuated, we divide this stage into two steps using the backstepping technique. At the first step, the robot heading ϕ_i and the robot linear velocity v_i are used as immediate controls to fulfill the task of position tracking and collision avoidance. At the second step, the robot angular velocity w_i is used as an immediate control to stabilize the error between the actual robot heading and its immediate value at the origin. We do not use the transformation in [14] to interpret the tracking errors in a frame attached to the reference trajectories as often done in literature (e.g. [15], [1], [16], [18]) to avoid difficulties when dealing with collision avoidance.

Step I.1

Define

$$\phi_{ie} = \phi_i - \alpha_{\phi_i},\ v_{ie} = v_i - \alpha_{v_i} \tag{6.7}$$

where α_{ϕ_i} and α_{v_i} are virtual controls of ϕ_i and v_i, respectively. With (6.7), the first two equations of (6.1) are read:

$$\dot{q}_i = u_i + \Delta_{1i} + \Delta_{2i} \tag{6.8}$$

where $q_i = [x_i\ \ y_i]^T$, and

$$\begin{aligned}
u_i &= \begin{bmatrix}\cos(\alpha_{\phi_i})\\ \sin(\alpha_{\phi_i})\end{bmatrix}\alpha_{v_i},\\
\Delta_{1i} &= \begin{bmatrix}(\cos(\phi_{ie})-1)\cos(\alpha_{\phi_i}) - \sin(\phi_{ie})\sin(\alpha_{\phi_i})\\ (\sin(\phi_{ie})\cos(\alpha_{\phi_i}) + (\cos(\phi_{ie})-1)\sin(\alpha_{\phi_i})\end{bmatrix}\alpha_{v_i},\\
\Delta_{2i} &= \begin{bmatrix}\cos(\phi_i)\\ \sin(\phi_i)\end{bmatrix}v_{ie}.
\end{aligned} \tag{6.9}$$

To fulfill the task of position tracking and collision avoidance, we consider the following potential function

$$\varphi_{I1} = \sum_{i=1}^{N}(\gamma_i + 0.5\beta_i) \tag{6.10}$$

where γ_i and β_i are the goal and related collision avoidance functions for the robot i specified as follows:

-The goal function is designed such that it puts penalty on the tracking error for the robot, and is equal to zero when the robot is at its desired position. A simple choice of this function is

$$\gamma_i = 0.5\|q_i - q_{id}\|^2. \tag{6.11}$$

-The related collision function β_i should be chosen such that it is equal to infinity whenever any robots come in contact with the robot i, i.e. a collision occurs, and attains the minimum value when the robot i is at its desired location with respect to other group members belong to the set $\mathbb{N}_i$ robots, where $\mathbb{N}_i$ is the set that contains all the robots in the group except for the robot i. This function is chosen as follows:

$$\beta_i = \sum_{j \in \mathbb{N}_i} \beta_{ij} \tag{6.12}$$

where the function $\beta_{ij} = \beta_{ji}$ is a function of $\|q_{ij}\|^2/2$ with $q_{ij} = q_i - q_j$, and enjoys the following properties:

$$\begin{aligned}
&1) \quad \beta_{ij} = 0,\ \beta'_{ij} = 0,\ \beta''_{ij} \geq 0 \quad \text{if} \quad \|q_{ij}\| = \|q_{ijd}\|,\\
&2) \quad \beta_{ij} > 0 \quad \text{if} \quad 0 < \|q_{ij}\| < b_{ij},\\
&3) \quad \beta_{ij} = 0,\ \beta'_{ij} = 0,\ \beta''_{ij} = 0,\ \beta'''_{ij} = 0 \quad \text{if} \quad \|q_{ij}\| \geq b_{ij},\\
&4) \quad \beta_{ij} = \infty \quad \text{if} \quad \|q_{ij}\| \leq (\underline{R}_i + \underline{R}_j),\\
&5) \quad \beta_{ij} \leq \mu_1,\ |\beta'_{ij}| \leq \mu_2,\ \text{and}\ |\beta''_{ij} q_{ij}^T q_{ij}| \leq \mu_3,\ \forall (\underline{R}_i + \underline{R}_j) < \|q_{ij}\| \leq b_{ij},\\
&6) \quad \beta_{ij} \text{ is at least three times differentiable with respect to}\\
&\quad\ \ \|q_{ij}\|^2/2 \text{ if } \|q_{ij}\| > (\underline{R}_i + \underline{R}_j)
\end{aligned} \tag{6.13}$$

where $q_{ijd} = q_{id} - q_{jd}$, b_{ij} is a strictly positive constant such that $(\underline{R}_i + \underline{R}_j) < b_{ij} \leq \min(\overline{R}_i, \overline{R}_j, \|q_{ijd}\|)$; μ_1, μ_2 and μ_3 are positive constants; $\beta'_{ij}, \beta''_{ij}$ and β'''_{ij} are defined as follows: $\beta'_{ij} = \infty, \beta''_{ij} = \infty, \beta'''_{ij} = \infty$ if $\|q_{ij}\| \leq (\underline{R}_i + \underline{R}_j)$; $\beta'_{ij} = \frac{\partial \beta_{ij}}{\partial(\|q_{ij}\|^2/2)}, \beta''_{ij} = \frac{\partial^2 \beta_{ij}}{\partial(\|q_{ij}\|^2/2)^2}, \beta'''_{ij} = \frac{\partial^3 \beta_{ij}}{\partial(\|q_{ij}\|^2/2)^3}$ if $\|q_{ij}\| > (\underline{R}_i + \underline{R}_j)$.

Remark 6.3. Properties 1) - 4) imply that the function β_i is positive definite, is equal to zero when all the robots are at their desired locations, and is equal to infinity when a collision between any robots in the group occurs. Moreover, Property 1) and the function γ_i given in (6.11) ensure that the function φ_{I1} attains the (unique) minimum value of zero when all the robots are at their desired positions. Also, Property 3) ensures that the collision avoidance between the robots i and j is only taken into account when they are in their communication areas. Property 5) is used to prove stability of the closed loop system. Property 6) ensures that we can use techniques such as the backstepping and direct Lyapunov design methods ([12], [58]) for control design and stability analysis for continuous systems instead of techniques for switched, nonsmooth and discontinuous systems ([59], [33]) to handle the collision avoidance problem under the robot limited sensing ranges.

There are many functions that satisfy all properties of β_{ij} given in (6.13). An example is

$$\beta_{ij} = \frac{h_{ij}(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2)}{1 - h_{ij}(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2)} \tag{6.14}$$

where $h_{ij}\left(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\right)$ is a p times differentiable bump function defined in Definition A.28 with $p \geq 3$ and $a_{ij} \geq (\underline{R}_i + \underline{R}_j)$, and $b_{ij} \leq \min\left(\overline{R}_i, \overline{R}_j, \|q_{ijd}\|\right)$.

The time derivative of φ_{I1} along the solutions of (6.8) satisfies

$$\dot{\varphi}_{I1} = \sum_{i=1}^{N} \Omega_i^T\left(u_i + \Delta_{1i} + \Delta_{2i} - \dot{q}_{od}\right) \tag{6.15}$$

where we have used $\dot{q}_{id} = \dot{q}_{od}, u_i - u_j = u_i - \dot{q}_{od} - (u_j - \dot{q}_{od}), \forall(i,j) \in (1,2,...,N), i \neq j$, and

$$\Omega_i = q_i - q_{id} + \sum_{j\in\mathbb{N}_i} \beta'_{ij} q_{ij}. \tag{6.16}$$

From (6.15), we choose a bounded control u_i designed as follows:

$$u_i = -k_0 u_{od}^2 \Psi(\Omega_i) + \dot{q}_{od} \tag{6.17}$$

where $\Psi(\Omega_i)$ denotes a vector of bounded functions of elements of Ω_i in the sense that $\Psi(\Omega_i) = \left[\psi(\Omega_{ix})\ \psi(\Omega_{iy})\right]^T$ with Ω_{ix} and Ω_{iy} the first and second rows of Ω_i, i.e. $\Omega_i = [\Omega_{ix}\ \Omega_{iy}]^T$. The function $\psi(x)$ is a scalar, at least three times differentiable and bounded function with respect to x, and satisfies

$$\begin{aligned}
&1)\ |\psi(x)| \leq \varrho_1, \\
&2)\ \psi(x) = 0 \quad \text{if } x = 0, \quad x\psi(x) > 0 \text{ if } x \neq 0, \\
&3)\ \psi(-x) = -\psi(x), (x-y)[\psi(x) - \psi(y)] \geq 0, \\
&4)\ |\psi(x)/x| \leq \varrho_2, |\partial\psi(x)/\partial x| \leq \varrho_3, \partial\psi(x)/\partial x|_{x=0} = 1
\end{aligned} \tag{6.18}$$

for all $x \in \mathbb{R}, y \in \mathbb{R}$, where $\varrho_1, \varrho_2, \varrho_3$ are strictly positive constants. Some functions that satisfy the above properties are $\arctan(x)$ and $\tanh(x)$. The strictly positive constant k_0 is chosen such that

$$k_0 < \frac{1}{2\varrho_1 u_{od}^{max}}. \tag{6.19}$$

The above condition ensures that α_{ϕ_i} and α_{v_i} are solvable from u_i. We now need to solve for α_{ϕ_i} and α_{v_i} from the expression of u_i in (6.17) and (6.9). From (6.17) and (6.9), we have

$$\begin{aligned}
\cos(\alpha_{\phi_i})\alpha_{v_i} &= -k_0 u_{od}^2 \psi(\Omega_{ix}) + \cos(\phi_{od}) u_{od}, \\
\sin(\alpha_{\phi_i})\alpha_{v_i} &= -k_0 u_{od}^2 \psi(\Omega_{iy}) + \sin(\phi_{od}) u_{od}
\end{aligned} \tag{6.20}$$

where we have used $\dot{x}_{od} = x'_{od}\dot{s}_{od} = \frac{x'_{od}\sqrt{x'^2_{od}+y'^2_{od}}\,\dot{s}_{od}}{\sqrt{x'^2_{od}+y'^2_{od}}} = \cos(\phi_{od})u_{od}$ and $\dot{y}_{od} = y'_{od}\dot{s}_{od} = \frac{y'_{od}\sqrt{x'^2_{od}+y'^2_{od}}\,\dot{s}_{od}}{\sqrt{x'^2_{od}+y'^2_{od}}} = \sin(\phi_{od})u_{od}$ since $\phi_{od} = \arctan(y'_{od}/x'_{od})$

and $\sqrt{x_{od}'^2 + y_{od}'^2} > 0$, see Assumption 6.1. The left hand sides of (6.20) are actually the coordinates of u_i in the x and y directions. Now multiplying both sides of the first equation of (6.20) with $\cos(\phi_{od})$ and both sides of the second equation of (6.20) with $\sin(\phi_{od})$ then adding the resulting equations together yield

$$\cos(\alpha_{\phi_i} - \phi_{od})\alpha_{v_i} = -k_0 u_{od}^2\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + u_{od}. \quad (6.21)$$

On the other hand multiplying both sides of the first equation of (6.20) with $\sin(\phi_{od})$ and both sides of the second equation of (6.20) with $\cos(\phi_{od})$ then subtracting the resulting equations give

$$\sin(\alpha_{\phi_i} - \phi_{od})\alpha_{v_i} = -k_0 u_{od}^2\big(-\psi(\Omega_{ix})\sin(\phi_{od}) + \psi(\Omega_{ix})\cos(\phi_{od})\big). \quad (6.22)$$

From (6.21) and (6.22), we solve for α_{ϕ_i} and α_{v_i} as

$$\begin{aligned}
\alpha_{\phi_i} &= \phi_{od} + \arctan\left(\frac{-k_0 u_{od}\big(-\psi(\Omega_{ix})\sin(\phi_{od}) + \psi(\Omega_{ix})\cos(\phi_{od})\big)}{-k_0 u_{od}\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + 1}\right),\\
\alpha_{v_i} &= \cos(\alpha_{\phi_i})\big(-k_0 u_{od}^2\psi(\Omega_{ix}) + \cos(\phi_{od})u_{od}\big) +\\
&\quad \sin(\alpha_{\phi_i})\big(-k_0 u_{od}^2\psi(\Omega_{iy}) + \sin(\phi_{od})u_{od}\big).
\end{aligned} \quad (6.23)$$

It is noted that (6.23) is well defined since

$$-k_0 u_{od}\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + 1 \geq -2\varrho_1 k_0 u_{od}^{max} + 1 > 0$$

where the condition (6.19) has been used. Moreover, it is of interest to note that α_{ϕ_i} and α_{v_i} are at least twice differentiable functions of $q_{od}, \phi_{od}, u_{od}, q_i, q_{ij}$ with $j \in \mathbb{N}_i, j \neq i$.

Remark 6.4. When Ω_i defined in (6.16) is substituted into (6.17), the control u_i can be written as

$$u_i = k_0 u_{od}^2 \begin{bmatrix} \psi\big(-(x_i - x_{id}) - \sum_{j\in\mathbb{N}_i}\beta'_{ij}(x_i - x_j)\big) \\ \psi\big(-(y_i - y_{id}) - \sum_{j\in\mathbb{N}_i}\beta'_{ij}(y_i - y_j)\big) \end{bmatrix} + \dot{q}_{od}. \quad (6.24)$$

It is seen from (6.24) that the argument of ψ consists of two parts. The first part, $-(x_i - x_{id})$ or $-(y_i - y_{id})$, referred to as the attractive force plays the role of forcing the robot i to its desired location. The second part, $-\sum_{j\in\mathbb{N}_i}\beta'_{ij}(x_i - x_j)$ or $-\sum_{j\in\mathbb{N}_i}\beta'_{ij}(y_i - y_j)$, referred to as the repulsive force, takes care of collision avoidance for the robot i with the other robots. Moreover, the immediate control u_i of the robot i given in (6.17) depends on only its own state and reference trajectory, and the states of other neighbor robots j if the points P_{0j} of these robots are in the circular communication area of the robot i, since outside this area $\beta'_{ij} = 0$, see Property 3) of β_{ij}.

Now substituting (6.17) into (6.15) results in

$$\dot{\varphi}_{I1} = -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) + \sum_{i=1}^{N} \Omega_i^T (\Delta_{1i} + \Delta_{2i}). \tag{6.25}$$

Substituting (6.17) into (6.8) results in

$$\dot{q}_i = -k_0 u_{od}^2 \Psi(\Omega_i) + \dot{q}_{od} + \Delta_{1i} + \Delta_{2i}. \tag{6.26}$$

Step I.2

At this step, we view w_i as an immediate control to stabilize ϕ_{ie} at the origin. As such, we define

$$w_{ie} = w_i - \alpha_{w_i} \tag{6.27}$$

where α_{w_i} is a virtual control of w_i. To prepare for design of the virtual control α_{w_i}, let us calculate $\dot{\phi}_{ie}$. Differentiating both sides of the first equation of (6.7) along the solutions of (6.27), the third equation of (6.1), and the second equation of (6.23) results in

$$\begin{aligned}\dot{\phi}_{ie} =& w_{ie} + \alpha_{w_i} - \frac{\partial \alpha_{\phi_i}}{\partial q_{od}} \dot{q}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial \phi_{od}} \dot{\phi}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial u_{od}} \dot{u}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial q_i} (u_i + \Delta_{1i} + \Delta_{2i}) - \\ & \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (u_i + \Delta_{1i} + \Delta_{2i} - (u_j + \Delta_{1j} + \Delta_{2j})).\end{aligned} \tag{6.28}$$

To design the virtual control α_{w_i}, we consider the following function

$$\varphi_{I2} = \varphi_{I1} + 0.5 \sum_{i=1}^{N} \phi_{ie}^2 \tag{6.29}$$

whose derivative along the solutions of (6.25) and (6.28) satisfies

$$\begin{aligned}\dot{\varphi}_{I2} = & -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) + \sum_{i=1}^{N} \Omega_i^T \Delta_{2i} + \sum_{i=1}^{N} \phi_{ie} \bigg(\frac{\Omega_i^T \Delta_{1i}}{\phi_{ie}} + w_{ie} + \\ & \alpha_{w_i} - \frac{\partial \alpha_{\phi_i}}{\partial q_{od}} \dot{q}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial \phi_{od}} \dot{\phi}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial u_{od}} \dot{u}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial q_i} (u_i + \Delta_{1i} + \Delta_{2i}) - \\ & \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (u_i + \Delta_{1i} + \Delta_{2i} - (u_j + \Delta_{1j} + \Delta_{2j})) \bigg).\end{aligned} \tag{6.30}$$

It is noted that $\frac{\Delta_{1i}}{\phi_{ie}}$ is well defined since $\frac{\sin(\phi_{ie})}{\phi_{ie}} = \int_0^1 \cos(\lambda \phi_{ie}) d\lambda$ and $\frac{\cos(\phi_{ie})-1}{\phi_{ie}} = \int_1^0 \sin(\lambda \phi_{ie}) d\lambda$ are smooth functions for all $\phi_{ie} \in \mathbf{R}$. From (6.30), we choose the virtual control α_{w_i} as

$$\alpha_{w_i} = -k_i\phi_{ie} - \frac{\Omega_i^T\Delta_{1i}}{\phi_{ie}} + \frac{\partial\alpha_{\phi_i}}{\partial q_{od}}\dot{q}_{od} + \frac{\partial\alpha_{\phi_i}}{\partial\phi_{od}}\dot{\phi}_{od} + \frac{\partial\alpha_{\phi_i}}{\partial u_{od}}\dot{u}_{od} + \frac{\partial\alpha_{\phi_i}}{\partial q_i}(u_i + \Delta_{1i}) + \sum_{j=1,j\neq i}^{N}\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}(u_i + \Delta_{1i} - (u_j + \Delta_{1j})) \quad (6.31)$$

where k_i is a positive constant. It is of interest to note that α_{w_i} is an at least once differentiable function of $q_{od}, \dot{q}_{od}, \phi_{od}, \dot{\phi}_{od}, u_{od}, \dot{u}_{od}, q_i, \phi_i, q_{ij}, \phi_j$ with $j \in \mathbb{N}_i, j \neq i$. Moreover, it should be noted that the virtual control α_{w_i} contains only the state and reference trajectory of the robot i, and the states of other neighbor robots j if the points P_{0j} of these robots are in the communication area of the robot i, because outside this area $\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}} = 0$ thanks to Property 3) of β_{ij}. Substituting (6.31) into (6.30) results in (after some manipulation):

$$\dot{\varphi}_{I2} = -k_0 u_{od}^2 \sum_{i=1}^{N}\Omega_i^T\Psi(\Omega_i) - \sum_{i=1}^{N}k_i\phi_{ie}^2 + \sum_{i=1}^{N}\Big[\phi_{ie}w_{ie} + \big(\Omega_i^T - \phi_{ie}\frac{\partial\alpha_{\phi_i}}{\partial q_i} - \sum_{j=1,j\neq i}^{N}\big(\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial\alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\big)\big)\Delta_{2i}\Big]. \quad (6.32)$$

Substituting (6.31) into (6.28) gives

$$\dot{\phi}_{ie} = -k_i\phi_{ie} - \frac{\Omega_i^T\Delta_{1i}}{\phi_{ie}} - \frac{\partial\alpha_{\phi_i}}{\partial q_i}\Delta_{2i} - \sum_{j=1,j\neq i}^{N}\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}(\Delta_{2i} - \Delta_{2j}) + w_{ie}. \quad (6.33)$$

To prepare for the next section, let us compute the term $\overline{M}_i\dot{\varpi}_{ie}$ where $\varpi_{ie} = [v_{ie}\ w_{ie}]^T$. From the second equation of (6.7), (6.27), and the last equation of (6.1), we have

$$\begin{aligned}\overline{M}_i\dot{\varpi}_{ie} &= -\overline{C}_i(w_i)\varpi_i - \overline{D}_i\varpi_i - \overline{M}_i[\dot{\alpha}_{v_i}\ \dot{\alpha}_{w_i}]^T + \overline{B}_i\tau_i \\ &= -\overline{D}_i\varpi_{ie} + \Phi_i\Theta_i + \overline{B}_i\tau_i\end{aligned} \quad (6.34)$$

where

$$\dot{\alpha}_{v_i} = \frac{\partial \alpha_{v_i}}{\partial q_{od}}\dot{q}_{od} + \frac{\partial \alpha_{v_i}}{\partial u_{od}}\dot{u}_{od} + \frac{\partial \alpha_{v_i}}{\partial \phi_{od}}\dot{\phi}_{od} + \frac{\partial \alpha_{v_i}}{\partial q_i}\vartheta_i + \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{v_i}}{\partial q_{ij}}(\vartheta_i - \vartheta_j),$$

$$\dot{\alpha}_{w_i} = \frac{\partial \alpha_{w_i}}{\partial q_{od}}\dot{q}_{od} + \frac{\partial \alpha_{w_i}}{\partial \dot{q}_{od}}\ddot{q}_{od} + \frac{\partial \alpha_{w_i}}{\partial u_{od}}\dot{u}_{od} + \frac{\partial \alpha_{w_i}}{\partial \dot{u}_{od}}\ddot{u}_{od} + \frac{\partial \alpha_{w_i}}{\partial \phi_{od}}\dot{\phi}_{od} + \frac{\partial \alpha_{w_i}}{\partial \dot{\phi}_{od}}\ddot{\phi}_{od} + \frac{\partial \alpha_{w_i}}{\partial q_i}\vartheta_i + \frac{\partial \alpha_{w_i}}{\partial \phi_i}w_i + \sum_{j=1, j\neq i}^{N}\left(\frac{\partial \alpha_{w_i}}{\partial \phi_j}w_j + \frac{\partial \alpha_{w_i}}{\partial q_{ij}}(\vartheta_i - \vartheta_j)\right),$$

$$\Phi_i = \begin{bmatrix} w_i^2 & -\alpha_{vi} & -\alpha_{wi} & -\dot{\alpha}_{vi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -v_i w_i - \alpha_{vi} & -\alpha_{wi} & -\dot{\alpha}_{wi} \end{bmatrix}, \tag{6.35}$$

$$\Theta_i = \begin{bmatrix} b_i c_i & \bar{d}_{11i} & \bar{d}_{12i} & \bar{m}_{11i} & c_i/b_i & \bar{d}_{21i} & \bar{d}_{22i} & \bar{m}_{22i} \end{bmatrix}^T$$

where $\vartheta_i = u_i + \Delta_{1i} + \Delta_{2i}, i = 1, ..., N$. Again, $\dot{\alpha}_{v_i}$ and $\dot{\alpha}_{w_i}$ contain only the state and reference trajectory of the robot i, and the states of other neighbor robots j if the points P_{0j} of these robots are in the communication area of the robot i, because outside this area $\frac{\partial \alpha_{v_i}}{\partial q_{ij}} = 0$, $\frac{\partial \alpha_{w_i}}{\partial q_{ij}} = 0$, and $\frac{\partial \alpha_{w_i}}{\partial \phi_j} = 0$ thanks to Property 3) of β_{ij}.

6.2.2 Stage II

At this stage, we design the actual control input vector τ_i and update laws for unknown system parameter vector Θ_i for each robot i. To do so, we consider the following function

$$\varphi_{II} = \varphi_{I2} + \frac{1}{2}\sum_{i=1}^{N}\left(\varpi_{ie}^T \overline{M}_i \varpi_{ie} + \tilde{\Theta}_i^T \Gamma_i^{-1}\tilde{\Theta}_i\right) \tag{6.36}$$

where $\tilde{\Theta}_i = \Theta_i - \hat{\Theta}_i$ with $\hat{\Theta}_i$ being an estimate of Θ_i, and Γ_i is a symmetric positive definite matrix. Differentiating both sides of (6.36) along the solutions of (6.34) and (6.32) yields

$$\dot{\varphi}_{II} = -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) - \sum_{i=1}^{N} k_i \phi_{ie}^2 + \sum_{i=1}^{N} \phi_{ie} w_{ie} + \sum_{i=1}^{N}\left(\Omega_i^T - \phi_{ie}\frac{\partial \alpha_{\phi_i}}{\partial q_i} - \sum_{j=1, j\neq i}^{N}\left(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\right)\right)\Delta_{2i} + \sum_{i=1}^{N}\left(\varpi_{ie}^T(-\overline{D}_i \varpi_{ie} + \Phi_i \Theta_i + \overline{B}_i \tau_i) - \tilde{\Theta}_i^T \Gamma_i^{-1}\dot{\hat{\Theta}}_i\right) \tag{6.37}$$

which suggests that we choose

$$\tau_i = \overline{B}_i^{-1}\Big(- L_i \varpi_{ie} - \Phi_i \hat{\Theta}_i - \Big[\Big(\Omega_i^T - \phi_{ie}\frac{\partial \alpha_{\phi_i}}{\partial q_i} - \sum_{j=1, j\neq i}^{N}\Big(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\Big)\Big)\overline{\Delta}_{2i} \quad \phi_{ie}\Big]^T\Big),$$
$$\dot{\hat{\Theta}}_i = \Gamma_i \Phi_i^T \varpi_{ie} \tag{6.38}$$

where $\overline{\Delta}_{2i} = [\cos(\phi_i) \ \ \sin(\phi_i)]^T$, and L_i is a symmetric positive definite matrix. Substituting the first equation of (6.38) into (6.34) gives

$$\overline{M}_i \dot{\varpi}_{ie} = -(\overline{D}_i + L_i)\varpi_{ie} + \Phi_i \tilde{\Theta}_i - \Big[\Big(\Omega_i^T - \phi_{ie}\frac{\partial \alpha_{\phi_i}}{\partial q_i} - \sum_{j=1, j\neq i}^{N}\Big(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\Big)\Big)\overline{\Delta}_{2i} \quad \phi_{ie}\Big]^T. \tag{6.39}$$

By construction, the control τ_i and the update $\dot{\hat{\Theta}}_i$ given in (6.38) of the robot i contain only the state and reference trajectory of the robot i, and the states of other neighbor robots j if these robots are in a circular area, which is centered at point P_{0i} of the robot i and has a radius no greater than $\overline{R}_i$. Now substituting (6.38) into (6.37) results in

$$\dot{\varphi}_{1i} = -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) - \sum_{i=1}^{N} k_i \phi_{ie}^2 - \sum_{i=1}^{N} \varpi_{ie}^T(\overline{D}_i + L_i)\varpi_{ie}. \tag{6.40}$$

For convenience, we rewrite the closed loop system consisting of (6.26), (6.33), (6.39), and the second equation of (6.38) as follows:

$$\begin{aligned}
\dot{q}_i &= -k_0 u_{od}^2 \Psi(\Omega_i) + \dot{q}_{od} + \Delta_{1i} + \Delta_{2i}, \\
\dot{\phi}_{ie} &= -k_i \phi_{ie} - \frac{\Omega_i^T \Delta_{1i}}{\phi_{ie}} - \frac{\partial \alpha_{\phi_i}}{\partial q_i}\Delta_{2i} - \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}(\Delta_{2i} - \Delta_{2j}) + w_{ie}, \\
\overline{M}_i \dot{\varpi}_{ie} &= -(\overline{D}_i + L_i)\varpi_{ie} + \Phi_i \tilde{\Theta}_i - \\
&\quad \Big[\Big(\Omega_i^T - \phi_{ie}\frac{\partial \alpha_{\phi_i}}{\partial q_i} - \sum_{j=1, j\neq i}^{N}\Big(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\Big)\Big)\overline{\Delta}_{2i} \quad \phi_{ie}\Big]^T, \\
\dot{\tilde{\Theta}}_i &= -\Gamma_i \Phi_i^T \varpi_{ie}.
\end{aligned} \tag{6.41}$$

We now state the main result of our paper in the following theorem.

Theorem 6.5. *Under Assumption 6.1, the control τ_i and the update law $\dot{\hat{\Theta}}_i$ given in (6.38) for the robot i solve the formation control objective. In particular, no collisions between any robots can occur for all $t \geq t_0 \geq 0$, the closed* *loop system (6.41) is forward complete, and the position and orientation of the robots track their reference trajectories asymptotically in the sense of (6.6).*

6.3 Proof of Theorem 6.5

We first prove that no collisions between the robots can occur, the closed loop system (6.41) is forward complete, and that the robots asymptotically approach their target points or some critical points. We then investigate stability of the closed loop system (6.41) at these points, and show that the position and orientation of the robots asymptotically track their reference trajectories.

+Proof of no collisions and complete forwardness of closed loop system. From (6.40) and properties of the function ψ , see (6.18), we have $\dot{\varphi}_{II} \leq 0$, which implies that $\varphi_{II}(t) \leq \varphi_{II}(t_0), \forall t \geq t_0$. With definition of the function φ_{II} in (6.36) and its components in (6.29), (6.10), (6.11) and (6.12), we have

$$\sum_{i=1}^{N}\Big[\gamma_i(t) + \frac{1}{2}\sum_{j\in\mathbb{N}_i}\beta_{ij}(t) + \frac{1}{2}\phi_{ie}(t) + \frac{1}{2}\varpi_{ie}^T(t)\overline{M}_i\varpi_{ie}(t) + \frac{1}{2}(\Theta_i - \hat{\Theta}_i(t))^T\Gamma_i^{-1}(\Theta_i - \hat{\Theta}_i(t))\Big] \leq \sum_{i=1}^{N}\Big[\gamma_i(t_0) + \frac{1}{2}\sum_{j\in\mathbb{N}_i}\beta_{ij}(t_0) + \frac{1}{2}\phi_{ie}(t_0) + \frac{1}{2}\varpi_{ie}^T(t_0)\overline{M}_i\varpi_{ie}(t_0) + \frac{1}{2}(\Theta_i - \hat{\Theta}_i(t_0))^T\Gamma_i^{-1}(\Theta_i - \hat{\Theta}_i(t_0))\Big] \tag{6.42}$$

for all $t \geq t_0 \geq 0$. From the condition specified in item 4) of Assumption 6.1, and Property 5) of β_{ij}, and definition of ϕ_{ie}, ϖ_{ie}, we have the right hand side of (6.42) is bounded by a positive constant depending on the initial conditions. Boundedness of the right hand side of (6.42) implies that the left hand side of (6.42) must be also bounded. As a result, $\beta_{ij}(t)$ must be smaller than some positive constant depending on the initial conditions for all $t \geq t_0 \geq 0$. From properties of β_{ij}, see (6.13), $\|q_{ij}(t)\| - (\underline{R}_i + \underline{R}_j)$ must be larger than some positive constant depending on the initial conditions denoted by ϵ_3, i.e. there are no collisions for all $t \geq t_0 \geq 0$. Boundedness of the left hand side of (6.42) also implies that of $(q_i(t) - q_{id}(t)), \phi_{ie}(t), \varpi_{ie}(t)$ and $\hat{\Theta}_i(t)$ for all $t \geq t_0 \geq 0$. This in turn implies by construction that $x_i(t), y_i(t), \phi_i(t), v_i(t)$ and $w_i(t)$ do not escape to infinity in finite time. Therefore, the closed loop system (6.41) is forward complete.

+Equilibrium points. Since we have already proved that there are no collisions between any robots, an application of Theorem 8.4 in [58] to (6.40) yields

$$\lim_{t\to\infty}\Big(k_0u_{od}^2(t)\sum_{i=1}^{N}\Omega_i^T(t)\Psi(\Omega_i(t))+\sum_{i=1}^{N}k_i\phi_{ie}^2(t)+\sum_{i=1}^{N}\varpi_{ie}^T(t)(\overline{D}_i+L_i)\varpi_{ie}(t)\Big)=0. \quad (6.43)$$

By noting that $\lim_{t\to\infty}u_{od}^2(t)\neq 0$ as specified in item 5) of Assumption 6.1, the limit equation (6.43) implies that

$$\lim_{t\to\infty}\Omega_i(t)=0,\ \lim_{t\to\infty}\phi_{ie}(t)=0,\ \lim_{t\to\infty}\varpi_{ie}(t)=0.$$

By construction, $\lim_{t\to\infty}\Omega_i(t)=0$ and $\lim_{t\to\infty}\phi_{ie}(t)=0$ imply that $\lim_{t\to\infty}(\phi_i(t)-\phi_{od}(t))=0$. Moreover, from definition of Ω_i in (6.16), $\lim_{t\to\infty}\Omega_i(t)=0$ means

$$\lim_{t\to\infty}\Big(q_i(t)-q_{id}(t)+\sum_{j\in\mathbb{N}_i}\beta'_{ij}q_{ij}(t)\Big)=0. \quad (6.44)$$

The limit equation (6.44) implies that $q(t)=[q_1^T(t)\ q_2^T(t),...,q_N^T(t)]^T$ can tend to $q_d=[q_{1d}^T\ q_{2d}^T,...,q_{Nd}^T]^T$ since $\beta'_{ij}|_{\|q_{ij}\|=\|q_{ijd}\|}=0$ (Property 1) of β_{ij}), or some vector denoted by $q_c=[q_{1c}^T\ q_{2c}^T,...,q_{Nc}^T]^T$ as the time goes to infinity, i.e. the equilibrium points can be q_d or q_c. It is noted that some elements of q_c can be equal to that of q_d. However, for simplicity we abuse the notation, i.e. we still denote that vector as q_c. Indeed, the vector q_c is such that

$$\Omega_i|_{q=q_c}=\Big[q_i-q_{id}+\sum_{j\in\mathbb{N}_i}\beta'_{ij}q_{ij}\Big]\Big|_{q=q_c}=0,\forall i=1,...,N. \quad (6.45)$$

To investigate properties of the equilibrium points, q_d and q_c, we consider the first equation of the closed loop system (6.41), i.e.

$$\dot{q}_i=-k_0u_{od}^2\Psi(\Omega_i)+\dot{q}_{od}+\Delta_{1i}+\Delta_{2i}. \quad (6.46)$$

Since we have already proved that the closed loop system (6.41) is forward complete, and $\lim_{t\to\infty}\phi_{ie}(t)=0$ and $\lim_{t\to\infty}\varpi_{ie}(t)=0$ imply from the expressions of Δ_{1i} and Δ_{2i}, see (6.9), that $\lim_{t\to\infty}(\Delta_{1i}(t)+\Delta_{2i}(t))=0$, we treat $\Delta_i(t)=\Delta_{1i}(t)+\Delta_{2i}(t)$ as an input to (6.46) instead of a state. Moreover, we have already proved that the trajectory, q, can approach either the set of desired equilibrium points denoted by q_d or the set of undesired equilibrium points denoted by q_c 'almost globally'. The term 'almost globally' refers to the fact that the agents start from a set that includes both condition (6.2) and that does not coincide at any point with the set of the undesired saddle point q_c. Therefore, we now need to prove that q_d is locally asymptotically stable and that q_c is locally unstable. Once this is proved, we can conclude that the trajectory q approaches q_d from almost everywhere except for from the set denoted by the condition (6.2) and the set denoted by q_c, which is unstable.

+Properties of equilibrium points. The system (6.46) can be written in a vector form as

$$\dot{q} = -k_0 u_{od}^2 \Psi_q(q, q_d) + \text{vec}(\dot{q}_{od}) + \text{vec}(\Delta_i) \tag{6.47}$$

where $\Psi_q(q, q_d) = [\Psi^T(\Omega_1), ..., \Psi^T(\Omega_i), ..., \Psi^T(\Omega_N)]^T$, $\text{vec}(\dot{q}_{od}) = [\dot{q}_{od}^T, ..., \dot{q}_{od}^T, ..., \dot{q}_{od}^T]^T$, and $\text{vec}(\Delta_i) = [\Delta_1^T, ..., \Delta_i^T, ..., \Delta_N^T]^T$. Therefore, near an equilibrium point q_o, which can be either q_d or q_c, we have

$$\dot{q} = -k_0 u_{od}^2 \, \partial\Psi_q(q, q_d)/\partial q\big|_{q=q_o}(q - q_o) + \text{vec}(\dot{q}_{od}) + \text{vec}(\Delta_i) \tag{6.48}$$

where the (i^{th}, j^{th}) element of the matrix $\frac{\partial\Psi_q(q,q_d)}{\partial q}$ is $\bigwedge_{ij} = \frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\frac{\partial\Omega_i}{\partial q_j}, (i,j) \in \mathbb{N}$ with $\mathbb{N}$ being the set of all agents. A simple calculation shows that

$$\frac{\partial\Omega_i}{\partial q_i} = \Big(1 + \sum_{i\in\mathbb{N}_i}\beta'_{ij}\Big)I_n + \sum_{j\in\mathbb{N}_i}\beta''_{ij}q_{ij}q_{ij}^T, \; \frac{\partial\Omega_i}{\partial q_j} = -\beta'_{ij}I_n - \beta''_{ij}q_{ij}q_{ij}^T. \tag{6.49}$$

for all $i = 1, ..., N, j \in \mathbb{N}_i, j \neq i$, where I_n denotes the identity matrix of size n. Let $\mathbb{N}^*$ be the set of the agents such that if the agents i and j belong to the set $\mathbb{N}^*$ then $\|q_{ij}\| < b_{ij}$. Next we will show that q_d is asymptotically stable and that q_c is unstable.

-Proof of q_d being asymptotic stable. Using properties of β_{ij} and ψ listed in (6.13) and (6.18), we have from (6.49) that for all $i = 1, ..., N, i \neq j$:

$$\begin{aligned}
&\frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\bigg|_{q=q_d} = I_n, \beta'_{ijd} = 0, \; \frac{\partial\Omega_i}{\partial q_i}\bigg|_{q=q_d} = I_n + \sum_{j\in\mathbb{N}_i}\beta''_{ijd}q_{ijd}q_{ijd}^T, \\
&\frac{\partial\Omega_i}{\partial q_j}\bigg|_{q=q_d} = -\beta''_{ijd}q_{ijd}q_{ijd}^T
\end{aligned} \tag{6.50}$$

where $\beta'_{ijd} = \beta'_{ij}\big|_{q_{ij}=q_{ijd}}$ and $\beta''_{ijd} = \beta''_{ij}\big|_{q_{ij}=q_{ijd}}$, with $q_{ijd} = q_{id} - q_{jd}$. We consider the function

$$V_d = \sqrt{1 + \|q - q_d\|^2} - 1 \tag{6.51}$$

whose derivative along the solutions of (6.48) with q_o replaced by q_d, using (6.50), and noting that $\dot{q}_{od} = \dot{q}_{id}$ satisfies

$$\dot{V}_d = \frac{1}{\sqrt{1+\|q-q_d\|^2}}\Big(-k_0u_{od}^2\sum_{i=1}^{N}\|q_i-q_{id}\|^2 -$$

$$k_0u_{od}^2 \sum_{(i,j)\in \mathbf{N}, i\neq j} \beta''_{ijd}\big(q_{ijd}^T(q_{ij}-q_{ijd})\big)^2 + \sum_{i=1}^{N}(q_i-q_{id})^T\Delta_i\Big)$$

$$\leq -\frac{k_0u_{od}^2}{\sqrt{1+\|q-q_d\|^2}}\sum_{i=1}^{N}\|q_i-q_{id}\|^2 + \sum_{i=1}^{N}\|\Delta_i\| \tag{6.52}$$

since $\beta''_{ijd}\geq 0$, see Property 1) in (6.13). The last inequality of (6.52) implies that q_d is asymptotically stable because $\lim_{t\to\infty}u_{od}^2(t)\neq 0$, and we have already proved that $\lim_{t\to\infty}\Delta_i(t)=0$.

- *Proof of q_c being unstable.* Again using properties of β_{ij} and ψ in (6.13) and (6.18), we have from (6.49) that

$$\left.\frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\right|_{q=q_c} = I_n,\ \left.\frac{\partial\Omega_i}{\partial q_i}\right|_{q=q_c} = \big(1+\sum_{j\in\mathbf{N}_i}\beta'_{ijc}\big)I_n + \sum_{j\in\mathbf{N}_i}\beta''_{ijc}q_{ijc}q_{ijc}^T,$$

$$\left.\frac{\partial\Omega_i}{\partial q_j}\right|_{q=q_c} = -\beta'_{ijc}-\beta''_{ijc}q_{ijc}q_{ijc}^T \tag{6.53}$$

for all $i=1,...,N, i\neq j$, where $q_{ijc}=q_{ic}-q_{jc}$, $\beta'_{ijc}=\beta'_{ij}\big|_{q_{ij}=q_{ijc}}$ and $\beta''_{ijc}=\beta''_{ij}\big|_{q_{ij}=q_{ijc}}$. Since the related collision avoidance functions β_i, see (6.12), are specified in terms of relative distances between agents and it is extremely difficult to obtain q_c explicitly by solving (6.45), it is very difficult to use the Lyapunov function candidate $V_c=0.5\|q-q_c\|$ to investigate stability of (6.48) at q_c. Therefore, we consider the Lyapunov function candidate

$$\bar{V}_c = \sqrt{1+\|\bar{q}-\bar{q}_c\|^2}-1 \tag{6.54}$$

where $\bar{q}=[q_{12}^T,q_{13}^T,...q_{1N}^T,q_{23}^T,...,q_{2N}^T,...,q_{N-1,N}^T]^T$ and $\bar{q}_c=[q_{12c}^T,q_{13c}^T,...q_{1Nc}^T,q_{23c}^T,...,q_{2Nc}^T,...,q_{N-1,Nc}^T]^T$. Differentiating both sides of (6.54) along the solution of (6.48) with q_o replaced by q_c gives

$$\dot{\bar{V}}_c = -\frac{k_0u_{od}^2}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbf{N}\setminus\mathbf{N}^*}\|q_{ij}-q_{ijc}\|^2 -$$

$$\frac{k_0u_{od}^2}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbf{N}^*}(1+N\beta'_{ijc})\|q_{ij}-q_{ijc}\|^2 -$$

$$\frac{k_0u_{od}^2N}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbf{N}^*}\beta''_{ijc}\big(q_{ijc}^T(q_{ij}-q_{ijc})\big)^2 +$$

$$\frac{1}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbf{N}}(q_{ij}-q_{ijc})^T(\Delta_i-\Delta_j) \tag{6.55}$$

where $i \neq j$ and (6.53) has been used. To investigate stability properties of $\bar{q}_c$ based on (6.55), we will use (6.45). Define $\Omega_{ijc} = \Omega_{ic} - \Omega_{jc}$, $\forall (i,j) \in \{1, ..., N\}$, $i \neq j$ where $\Omega_{ic} = \Omega_i|_{q=q_c} = 0$, see (6.45). Therefore $\Omega_{ijc} = 0$. Hence $\sum_{(i,j)\in \mathbb{N}^*} q_{ijc}^T \Omega_{ijc} = 0, i \neq j$, which by using (6.45) is expanded to

$$\sum_{(i,j)\in \mathbb{N}^*} \left(q_{ijc}^T(q_{ijc} - q_{ijd}) + N\beta'_{ijc} q_{ijc}^T q_{ijc}\right) = 0$$
$$\Rightarrow \sum_{(i,j)\in \mathbb{N}^*} (1 + N\beta'_{ijc}) q_{ijc}^T q_{ijc} = \sum_{(i,j)\in \mathbb{N}^*} q_{ijc}^T q_{ijd} \tag{6.56}$$

where $i \neq j$. The sum $\sum_{(i,j)\in \mathbb{N}^*} q_{ijc}^T q_{ijd}$ is strictly negative since at the point, say F, where $q_{ij} = q_{ijd}$, $\forall (i,j) \in \mathbb{N}^*, i \neq j$ all attractive and repulsive forces are equal to zero while at the point, say C, where $q_{ij} = q_{ijc}$ $\forall (i,j) \in \mathbb{N}^*, i \neq j$ the sum of attractive and repulsive forces are equal to zero (but attractive and repulsive forces are nonzero). Therefore the point, say O, where $q_{ij} = 0$, $\forall (i,j) \in \mathbb{N}^*, i \neq j$ must locate between the points F and C for all $(i,j) \in \mathbb{N}^*, i \neq j$. That is there exists a strictly positive constant b such that $\sum_{(i,j)\in \mathbb{N}^*} q_{ijc}^T q_{ijd} < -b$, which is substituted into (6.56) to yield

$$\sum_{(i,j)\in \mathbb{N}^*} (1 + N\beta'_{ijc}) q_{ijc}^T q_{ijc} < -b, i \neq j. \tag{6.57}$$

Since $q_{ijc}^T q_{ijc} > 0, \forall (i,j) \not\in \mathbb{N}^*, i \neq j$, there exists a nonempty set $\mathbb{N}^{**} \subset \mathbb{N}^*$ such that for all $(i,j) \in \mathbb{N}^{**}, i \neq j$, $(1 + N\beta'_{ijc})$ is strictly negative, i.e. there exists a strictly positive constant b^{**} such that $(1 + N\beta'_{ijc}) < -b^{**}$, $\forall (i,j) \in \mathbb{N}^{**}, i \neq j$. We now write (6.55) as

$$\begin{aligned}
\dot{V}_c = &-\frac{k_0 u_{od}^2}{\sqrt{1 + \|\bar{q} - \bar{q}_c\|^2}} \Bigg[\sum_{(i,j)\in \mathbb{N}\setminus \mathbb{N}^*} \|q_{ij} - q_{ijc}\|^2 + \\
&\sum_{(i,j)\in \mathbb{N}^*\setminus \mathbb{N}^{**}} (1 + N\beta'_{ijc}) \|q_{ij} - q_{ijc}\|^2 + \\
&N \sum_{(i,j)\in \mathbb{N}^*} \beta''_{ijc} \left(q_{ijc}^T (q_{ij} - q_{ijc})\right)^2 \Bigg] - \\
&\frac{k_0 u_{od}^2}{\sqrt{1 + \|\bar{q} - \bar{q}_c\|^2}} \sum_{(i,j)\in \mathbb{N}^{**}} (1 + N\beta'_{ijc}) \|q_{ij} - q_{ijc}\|^2 + \\
&\frac{1}{\sqrt{1 + \|\bar{q} - \bar{q}_c\|^2}} \sum_{(i,j)\in \mathbb{N}} (q_{ij} - q_{ijc})^T (\Delta_i - \Delta_j)
\end{aligned} \tag{6.58}$$

where $i \neq j$. We now define a subspace such that $q_{ij} - q_{ijc} = \hat{0}$, $\forall (i,j) \in \mathbb{N} \setminus \mathbb{N}^{**}$ and $q_{ijc}^T (q_{ij} - q_{ijc}) = 0$, $\forall (i,j) \in \mathbb{N}^*, i \neq j$. In this subspace, we have

$$\begin{aligned}
\bar{V}_c &= \sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2} - 1, \\
\dot{\bar{V}}_c &= -\frac{k_0 u_{od}^2 \sum_{(i,j)\in\mathbb{N}^{**}} (1 + N\beta'_{ijc})\|q_{ij} - q_{ijc}\|^2}{\sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2}} + \\
&\quad \frac{\sum_{(i,j)\in\mathbb{N}^{**}} (q_{ij} - q_{ijc})^T (\Delta_i - \Delta_j)}{\sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2}}
\end{aligned} \tag{6.59}$$

Using $(1 + N\beta'_{ijc}) < -b^{**}$, $\forall (i,j) \in \mathbb{N}^{**}, i \neq j$, we can write (6.59) as

$$\begin{aligned}
\bar{V}_c &= \sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2} - 1, \\
\dot{\bar{V}}_c &\geq b^{**} k_0 u_{od}^2 \bar{V}_c - \sum_{(i,j)\in\mathbb{N}^{**}} \|\Delta_i - \Delta_j\|.
\end{aligned} \tag{6.60}$$

Now assume that q_c is a stable equilibrium point, i.e. $\lim_{t\to\infty} \|q_i(t) - q_{ic}\| = d_i, \forall i \in \mathbb{N}$ with d_i a nonnegative constant. Note that $\mathbb{N}^{**} \subset \mathbb{N}$, we have $\lim_{t\to\infty} \|q_i(t) - q_{ic}\| = d_i, \forall i \in \mathbb{N}^{**}$, which implies $\lim_{t\to\infty} \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}, \forall (i,j) \in \mathbb{N}^{**}, i \neq j$ with d^{**} a nonnegative constant, since $q_{ij} = q_i - q_j$ and $q_{ijc} = q_{ic} - q_{jc}$. We now consider two cases: $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| \neq 0$ and $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| = 0$.

Case I: $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| \neq 0$. Since $\lim_{t\to\infty} u_{od}^2(t) \neq 0$ (Assumption 1) and we have already shown that $\lim_{t\to\infty} \Delta_i(t) = 0, \forall i \in \mathbb{N}$, $\bar{V}_c$ in (6.60) is divergent. Therefore, $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\|$ cannot tend to a constant but must be divergent. This contradicts $\lim_{t\to\infty} \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}$, i.e. q_c cannot be a set of stable equilibrium points but must be a set of an unstable ones in this case.

Case II: $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| = 0$. There would be no contradiction. However this case is never observed in practice since the ever-present physical noise would cause $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\|$ to be different from 0 at the time $t^* > t_0$. We now need to show that once the sum $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\|$ is different from zero, this sum will not come back zero again for all $t \geq t^*$, i.e. the set of undesired equilibrium points q_c is not attractive. To do so, consider (6.60) with the initial time t^* instead of t_0, i.e.

$$\begin{aligned}
\bar{V}_c(t) &= \sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\|^2} - 1, \\
\dot{\bar{V}}_c(t) &\geq b^{**} k_0 u_{od}^2 \bar{V}_c(t) - \sum_{(i,j)\in\mathbb{N}^{**}} \|\Delta_i(t) - \Delta_j(t)\|
\end{aligned} \tag{6.61}$$

for $t \geq t^*$ and $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\| \geq \delta^*$, where δ^* is a positive constant. Since $\lim_{t\to\infty} u_{od}^2(t) \neq 0$ (Assumption 1) and we have already shown that $\lim_{t\to\infty} \Delta_i(t) = 0, \forall i \in \mathbb{N}$, $\bar{V}_c$ in (6.61) is divergent for $t \geq t^*$. Therefore, $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\|$, for $t \geq t^*$, cannot tend to a constant but must be divergent. This contradicts $\lim_{t\to\infty} \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}$, i.e. q_c must also be a set of unstable ones point in this case. Proof of Theorem 6.5 is completed. □

6.4 Simulations

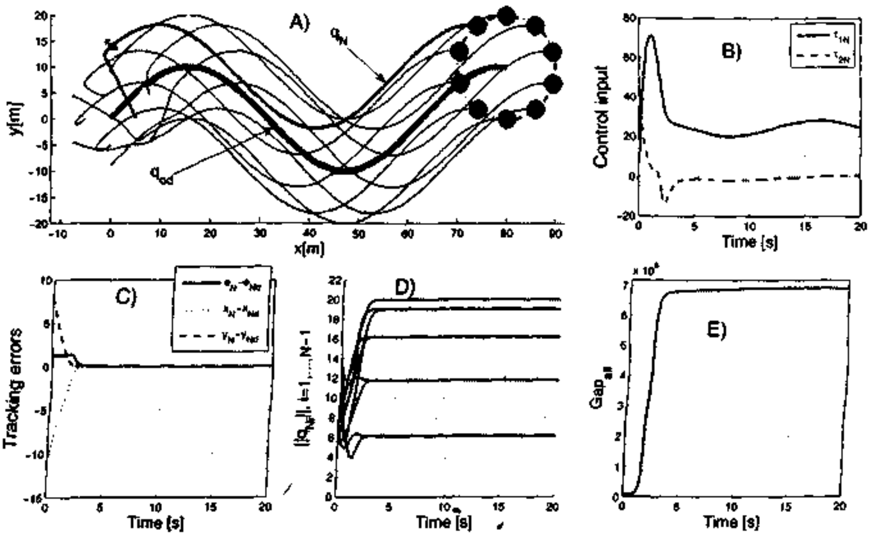


Fig. 6.3. Simulation results with 10 robots.

In this section, we simulate formation control of a group of $N = 10$ mobile robots to illustrate the effectiveness of the proposed controller. The physical parameters are taken from Section 1.3.1, Chapter 1, and $\underline{R}_i = 1.8, \overline{R}_i = 5, i = 1,...,N$. The robots are initialized as follows: $x_i = R_0 \sin((i-1)2\pi/6), y_i = R_0 \cos((i-1)2\pi/6), \omega_{1i} = 0, \omega_{2i} = 0$, where $R_0 = 9$ for $i = 1,...,6$ and $R_0 = 5$ for $i = 7,...,N$. The initial values of $\phi_i, i = 1,...,N$ are taken as random numbers between 0 and 2π. The initial values of $\hat{\Theta}_i$ are taken as half of their true values. The reference trajectories are taken as $q_{od} = [s \;\; 10\sin(0.1s)]^T, \dot{s} = 2$ and $l_i = 10[\sin((i-1)2\pi/N) \;\; \cos((i-1)2\pi/N)]^T$. This choice of the reference trajectories means that the common reference trajectory q_{od} forms a sinusoidal trajectory, and that the desired formation configuration is a polygon whose vertices uniformly distribute on a circle centered on the common reference trajectory and with a radius 10. The functions $\beta_{ij}, (i,j) \in \mathbb{N}, i \neq j$ are taken as in (6.14) with $p = 4, a_{ij} = 2\underline{R}_i, b_{ij} = \overline{R}_i$. The function $\psi()$ is taken as arctan(). The control gains and update gains are chosen as $k_0 = 0.1, k_i = 2, L_i = 4I_2, \Gamma_i = 0.2I_6$, where I_z is a z dimensional identity matrix. Indeed, the above choice of k_0 satisfies condition (6.19). Simulation results are plotted in Fig.6.3. It is seen that all robots nicely track their reference trajectories. During the first four seconds, the robots quickly move away from each other to avoid collisions then track their desired reference trajectory, see sub-figure A) in Fig.6.3, where the trajectory of the robot N is plotted in the thick line. For clarity, only the control inputs $[\tau_{1N} \;\; \tau_{2N}]^T$ of the robot N are plotted in sub-figure B) of Fig.6.3. Sub-figure C) in Fig.6.3 plots the tracking errors $x_N - x_{Nd}, y_N - y_{Nd}, \phi_N - \phi_{Nd}$ of the robot N. Indeed these errors tend to zero asymptotically. The distances between the robot N and other robots are plotted in sub-figure D) of Fig.6.3. Clearly, these distances are always larger than $\underline{R}_N + \underline{R}_i = 3.6, i = 1, ..., N-1$, i.e. there are no collisions between the robot N and all other robots in the group. Moreover, in sub-figure E) we plot product of all gaps between robots: $Gap_{all} = \prod_{(i,j)\in\mathbb{N}, i\neq j}(\|q_{ij}\| - (R_i + R_j))$. It is seen that Gap_{all} is always larger than zero. This means that $(\|q_{ij}\| - (R_i + R_j) > 0, \forall (i,j) \in \mathbb{N}, t \geq 0$, i.e. no collisions between any agents occurred. For clarity, we only plot the results for the first 20 seconds in sub-figures B), C), D) and E).

6.5 Notes and references

Decentralized navigation of non-point agents with single integrator dynamics was also investigated in [60] but each agent requires global knowledge of position of other agents. Formation control of vehicles with nonholonomic constraints was also considered in the literature(e.g. [40]). However, the nonholonomic kinematics are transformed to a double integrator dynamics by controlling the hand position instead of the inertial position of the vehicles. Consequently, the vehicle heading is not controlled. In addition, switching control theory [59] is often used to design a decentralized formation control system (e.g. [39], where a case by case basis is proposed), especially when the vehicles have limited sensing ranges and collision avoidance between vehicles must be considered. Clearly, it is more desirable if we are able to design a non-switching formation control system that can handle the above decentralized and collision avoidance requirements. Moreover, in the tracking control of single nonholonomic mobile robots (e.g. [15], [1], [18], [16]), where it seems that the backstepping technique was first used in [15] to take the robot kinetic into account in the control design, the tracking errors are often interpreted in a frame attached to the reference trajectory using the coordinate transformation in [14] to overcome difficulties due to nonholonomic constraints. If these techniques are migrated to formation control of a group of mobile robots, it is extremely difficult to incorporate collision avoidance between the robots. The above problems motivated the contribution of this chapter. The material in this chapter is based on the work in [61].

7

Formation Control of Mobile Robots with Limited Sensing: Output Feedback

Based on the material presented in Chapters 5 and 6, this chapter presents a constructive method to design output-feedback cooperative controllers that force a group of N unicycle-type mobile robots with limited sensing ranges to perform desired formation tracking, and guarantee no collisions between the robots. The robot velocities are not required for control implementation. For each robot an interlaced observer, which is a reduced order observer plus an interlaced term, is designed to estimate the robot unmeasured velocities. The observer design is based on a coordinate transformation that transforms the robot dynamics to a new dynamics, which does not contain velocity quadratic terms. The interlaced term is determined after the formation control design is completed to avoid difficulties due to observer errors and consideration of collision avoidance. Smooth and p times differentiable jump functions given in Section A.7 are incorporated into novel potential functions to design a formation tracking control system. Despite the robot limited sensing ranges, no switchings are needed to solve the collision avoidance problem. Simulations illustrate the results.

7.1 Problem statement

7.1.1 Robot dynamics

We consider a group of N mobile robots, of which each has the following dynamics (see Chapter 1, Section 1.3.1):

$$\begin{aligned}
\dot{x}_i &= v_i \cos(\phi_i) \\
\dot{y}_i &= v_i \sin(\phi_i) \\
\dot{\phi}_i &= w_i \\
\overline{M}_i \dot{\varpi}_i &= -\overline{C}_i(w_i)\varpi_i - \overline{D}_i \varpi_i + \overline{B}_i \tau_i, i = 1, ..., N
\end{aligned} \tag{7.1}$$

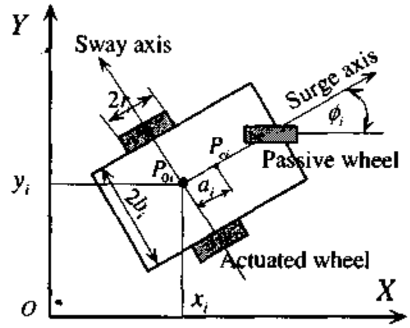


Fig. 7.1. Robot parameters.

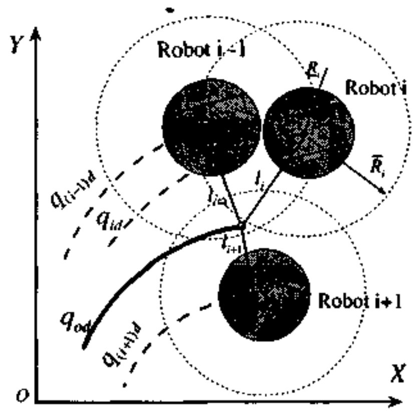


Fig. 7.2. Formation setup.

where all the state variables and parameters are defined in Section 1.3.1, Chapter 1.

7.1.2 Formation control objective

In this chapter, we will study a formation control problem under the following assumption:

Assumption 7.1 *1. The robot velocities (ω_{1i} and ω_{2i}) or (v_i and w_i) of each robot i are not available for control design.*

2. The robot i, see Figure 7.1 has a physical safety circular area, which is centered at the point P_{0i} and has a radius $\underline{R}_i$, and has a circular communication area, which is centered at the point P_{0i} and has a radius $\overline{R}_i$, see Fig. 7.2. The radius $\overline{R}_i$ is strictly larger than $\underline{R}_i + \underline{R}_j$, $j = 1, ..., N, j \neq i$.

3. *The robot i broadcasts its signals, $(x_i, y_i, \phi_i, \hat{\varpi}_i)$, where $\hat{\varpi}_i$ is an estimate of ϖ_i to be designed later, and its reference trajectory, q_{id}, in its communication area. Moreover, the robot i can receive the signals, $(x_j, y_j, \phi_j, \hat{\varpi}_j)$, and reference trajectories, q_{jd} broadcasted by other robots $j, j = 1, ..., N, j \neq i$ in the group if the points P_{0j} of these robots are in the communication area of the robot i.*
4. *At the initial time $t_0 \geq 0$, each robot starts at a location outside of the safety areas of other robots in the group, i.e. there exists a strictly positive constant ε_1 such that*

$$\|q_i(t_0) - q_j(t_0)\| - (\underline{R}_i + \underline{R}_j) \geq \varepsilon_1, \ \forall (i,j) \in (1, 2..., N), \ i \neq j \tag{7.2}$$

where $q_i = [x_i \ y_i]^T$.

5. *The reference trajectory for the robot i is $q_{id} = [x_{id} \ y_{id}]^T$, which is generated by*

$$q_{id} = q_{od}(s_{od}) + l_i \tag{7.3}$$

where $q_{od}(s_{od}) = [x_{od}(s_{od}) \ y_{od}(s_{od})]^T$ is referred to as the common reference trajectory with s_{od} being the common trajectory parameter, and l_i is a constant vector. The trajectory q_{od} satisfies the following conditions

$$\begin{aligned} &\lim_{t\to\infty} u_{od}^2(t) \neq 0, \ u_{od} = \sqrt{x_{od}'^2 + y_{od}'^2} \\ &\dot{s}_{od}, \sqrt{x_{od}'^2 + y_{od}'^2} > 0, \ |u_{od}(t)| \leq u_{od}^{max} \end{aligned} \tag{7.4}$$

where $x'_{od} = \frac{\partial x_{od}}{\partial s_{od}}$, $y'_{od} = \frac{\partial y_{od}}{\partial s_{od}}$, and u_{od}^{max} is a strictly positive constant. Moreover, $\dot{u}_{od}, \ddot{u}_{od}$ are also bounded. The constant vectors $l_i, i = 1, 2, ..., N$ satisfy

$$\|l_i - l_j\| - (\underline{R}_i + \underline{R}_j) \geq \varepsilon_2, \ \forall (i,j) \in (1, 2..., N), \ i \neq j \tag{7.5}$$

where ε_2 is a strictly positive constant.

Remark 7.2. Item 1) in Assumption 7.1 implies that we need to consider an output feedback control problem. Items 2) and 3) in Assumption 7.1 specify the way each robot communicates with other robots in the group within its communication range. In Fig. 7.2, the robots i and $i-1$ are communicating with each other since the points $P_{0,i-1}$ and P_{0i} are in the communication areas of the robots i and $i-1$, respectively. The robots i and $i+1$ are not communicating with each other since the points P_{0i} and $P_{0,i+1}$ are not in the communication areas of the robots $i+1$ and i, respectively. Similarly, the robots $i-1$ and $i+1$ are not communicating with each other. In item 5), the constant vectors $l_i, i = 1, ..., N$ specifies the desired formation configuration with respect to the earth-fixed frame OXY. The condition (7.4) implies that the common reference q_{od} is regular and its velocity u_{od}, which specifies how the desired formation, whose configuration is determined by l_i, moves along q_{od}, is bounded and satisfies a persistent excitation condition, i.e. the desired formation always moves along the common reference trajectory, q_{od}. The condition (7.5) specifies feasibility of the reference trajectories $q_{id}, i = 1, 2, ..., N$ (recall from (7.3) that $q_{id} - q_{jd} = l_i - l_j, \forall i \neq j$) due to physical safety circular areas of the robots. Finally, all the robots in the group require knowledge of the common reference trajectory q_{od} since this trajectory specifies how the whole formation should move with respect to the earth-fixed frame OXY.

Formation control objective: Under Assumption 7.1, for each robot i design an observer to estimate its velocities and the control input τ_i such that each robot asymptotically tracks its desired reference trajectory q_{id} while avoids collisions with all other robots in the group, i.e. for all $(i,j) \in \{1, 2, ..., N\}, i \neq j, t \geq t_0 \geq 0$

$$\begin{aligned} &\lim_{t\to\infty}(q_i(t) - q_{id}(t)) = 0, \\ &\lim_{t\to\infty}(\phi_i(t) - \phi_{id}(t)) = 0, \\ &\|q_i(t) - q_j(t)\| - (\underline{R}_i + \underline{R}_j) \geq \epsilon_3 \end{aligned} \tag{7.6}$$

where $\phi_{id}(t) = \arctan(y'_{od}/x'_{od})$, and ϵ_3 is a positive constant.

7.2 Observer design

For the sake of self-containing, we repeat the coordinate transformation for the observer design in Chapter 2, Section 2.2.2. It is noted that we will not use the same observer as in Section 2.2.2 but will design an interlaced observer to overcome the obstacle caused by the collision avoidance objective. As such, we first transform the dynamics (7.1) of the robot i to a new dynamics that does not contain the quadratic term $\overline{C}_i(w_i)\varpi_i$. As such, we introduce the following coordinate transformation:

$$X_i = Q_i(\eta_i)\varpi_i \tag{7.7}$$

where $\eta_i = [x_i \; y_i \; \phi_i]^T$, and $Q_i(\eta_i)$ is an invertible matrix to be determined. Differentiating both sides of (7.7) along the solutions of the first three equations of (7.1) results in

$$\dot{X}_i = \left[\dot{Q}_i(\eta_i)\varpi_i - Q_i(\eta_i)\overline{M}_i^{-1}\overline{C}_i(w_i)\varpi_i\right] + Q_i(\eta_i)\overline{M}_i^{-1}(-\overline{D}_i\varpi_i + \overline{B}_i\tau_i). \tag{7.8}$$

Our goal at this moment is to find the matrix $Q_i(\eta_i)$ so that

$$\left[\dot{Q}_i(\eta_i)\varpi_i - Q_i(\eta_i)\overline{M}_i^{-1}\overline{C}_i(w_i)\varpi_i\right] = 0, \forall(\eta_i, \varpi_i) \in \mathbb{R}^5. \tag{7.9}$$

Using the first three equations of (7.1), the above equation is equivalent to

$$\begin{aligned}
&\frac{\partial q_{ik1}}{\partial x_i}\cos(\phi_i) + \frac{\partial q_{ik1}}{\partial y_i}\sin(\phi_i) = 0,\\
&\frac{\partial q_{ik2}}{\partial \phi_i} + \frac{b_i c_i}{\bar{m}_{11i}} q_{ik1} = 0,\\
&\frac{\partial q_{ik1}}{\partial \phi_i} + \frac{\partial q_{ik2}}{\partial x_i}\cos(\phi_i) + \frac{\partial q_{ik2}}{\partial y_i}\sin(\phi_i) - \frac{c_i}{b_i\bar{m}_{22i}} q_{ik2} = 0
\end{aligned} \tag{7.10}$$

where q_{ikl} with $k = 1, 2$ and $l = 1, 2$ denote the components of the matrix $Q_i(\eta_i)$. The above set of partial differential equations has the following family of solutions

$$\begin{aligned}
q_{ik1} &= C_{ik1}\sin(c_i\Delta_i\phi_i) + C_{ik2}\cos(c_i\Delta_i\phi_i),\\
q_{ik2} &= b_i\bar{m}_{22i}\Delta_i\big(C_{ik1}\cos(c_i\Delta_i\phi_i) - C_{ik2}\sin(c_i\Delta_i\phi_i)\big)
\end{aligned} \tag{7.11}$$

where $\Delta_i = \frac{1}{\sqrt{\bar{m}_{11i}\bar{m}_{22i}}}$, and C_{ik1} and C_{ik2} are arbitrarily constants. Choosing $C_{i11} = C_{i22} = 0$, $C_{i12} = C_{i21} = 1$ results in

$$Q_i(\eta_i) = \begin{bmatrix} \cos(c_i\Delta_i\phi_i) & -b_i\Delta_i\bar{m}_{22i}\sin(c_i\Delta_i\phi_i) \\ \sin(c_i\Delta_i\phi_i) & b_i\Delta_i\bar{m}_{22i}\cos(c_i\Delta_i\phi_i) \end{bmatrix}. \tag{7.12}$$

Indeed, $Q_i(\eta_i)$ given in (7.12) is invertible for all $\phi_i \in \mathbb{R}$ since $\det(Q_i(\eta_i)) = b_i\Delta_i\bar{m}_{22i}$. Moreover, all elements of $Q_i(\eta_i)$ are bounded by a constant for all $\phi_i \in \mathbb{R}$. Now, substituting (7.12) into (7.8) results in

$$\dot{X}_i = Q_i(\eta_i)\overline{M}_i^{-1}(-\overline{D}_i Q_i^{-1}(\eta_i)X_i + \overline{B}_i\tau_i) \tag{7.13}$$

where we have used $\varpi_i = Q_i^{-1}(\eta_i)X_i$. From (7.13) we design a reduced observer to estimate X_i as follows:

$$\dot{\hat{X}}_i = Q_i(\eta_i)\overline{M}_i^{-1}(-\overline{D}_i Q_i^{-1}(\eta_i)\hat{X}_i + \overline{B}_i\tau_i) + \Xi_i \tag{7.14}$$

where $\hat{X}_i$ is an estimate of X_i, and Ξ_i is an interlaced term to be specified in the control design. This interlaced term is included in (7.14) to overcome difficulties due to the observer errors in the control design in the next section. The idea of introducing the interlaced term Ξ_i in the observer (7.14) is illustrated in the following simple example.

Example 7.3. Consider the nonlinear system:

$$\dot{x}_1 = f_1(x_1)x_2, \quad \dot{x}_2 = -x_2 + u \tag{7.15}$$

where x_1, x_2 are the states, u is the control input, and $f_1(x_1)$ is a nonlinear function of x_1 with assumption that $f_1(x_1) \neq 0, \forall x_1 \in \mathbb{R}$. The objective is to design the control u such that the system (7.15) is stabilized at the origin without measuring x_2. We first design a reduced order observer as follows:

$$\dot{\hat{x}}_2 = -\hat{x}_2 + u + \vartheta \tag{7.16}$$

where ϑ is an interlaced term to be determined later. Let the observer error be $\tilde{x}_2 = x_2 - \hat{x}_2$, then from (7.16) and the second equation of (7.15), we have

$$\dot{\tilde{x}}_2 = -\tilde{x}_2 - \vartheta. \tag{7.17}$$

Using the backstepping technique [12], we design the control input u for the system, which consists of the first equation of (7.15) and (7.16):

$$\dot{x}_1 = f_1(x_1)(\hat{x}_2 + \tilde{x}_2), \quad \dot{\hat{x}}_2 = -\hat{x}_2 + u + \vartheta \tag{7.18}$$

as follows:

$$\begin{aligned} & x_{1e} = x_1, \quad x_{2e} = \hat{x}_2 - \alpha_1, \quad \alpha_1 = -\frac{k_1 x_{1e}}{f_1(x_1)} \\ \Rightarrow \quad & \begin{cases} \dot{x}_{1e} = -k_1 x_{1e} + f_1(x_1)x_{2e} + f_1(x_1)\tilde{x}_2 \\ \dot{x}_{2e} = -\hat{x}_2 + u + \vartheta - \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1)\hat{x}_2 + f_1(x_1)\tilde{x}_2) \end{cases} \end{aligned} \tag{7.19}$$

where k_1 is a positive constant. We now take the Lyapunov function candidate: $V = \frac{1}{2}(x_{1e}^2 + x_{2e}^2 + \tilde{x}_2^2)$ whose derivative along the solutions of (7.19) and (7.17) is

$$\begin{aligned} \dot{V} &= -k_1 x_{1e}^2 - \tilde{x}_2^2 + x_{2e}\Big(f_1(x_1)x_{1e} - \hat{x}_2 + u + \vartheta - \frac{\partial \alpha_1}{\partial x_1} f_1(x_1)\hat{x}_2\Big) + \\ & \tilde{x}_2\Big(- \vartheta + x_{1e}f_1(x_1) - x_{2e}\frac{\partial \alpha_1}{\partial x_1} f_1(x_1)\Big) \end{aligned} \tag{7.20}$$

which suggest that we choose the interlaced term as: $\vartheta = x_{1e}f_1(x_1) - x_{2e}\frac{\partial \alpha_1}{\partial x_1} f_1(x_1)$, and the control as: $u = -k_2 x_{2e} - f_1(x_1)x_{1e} + \hat{x}_2 - \vartheta + \frac{\partial \alpha_1}{\partial x_1} f_1(x_1)\hat{x}_2$ where k_2 is a positive constant. With this choice, we have $\dot{V} = -k_1 x_{1e}^2 - k_2 x_{2e}^2 - \tilde{x}_2^2$. This implies that $x_{1e}(t), x_{2e}(t)$ and $\tilde{x}_2(t)$ exponentially converge to zero from any initial conditions. This simple example illustrates the idea of including the interlaced term ϑ in the observer (7.16). This interlaced term is then determined in the control design to void using nonlinear damping terms as in [12]. Nonlinear damping terms are usually used to handle nonlinearities coupling with observer errors. However, nonlinear damping terms introduce strong nonlinearities in the (virtual/actual) control. Moreover, in our formation control problem we cannot use nonlinear damping terms to handle the observer errors, see Section 7.3. Hence we proposed the interlaced term Ξ_i in the observer (7.14). □

We now return to the observer design. Define the observer error $\tilde{X}_i = X_i - \hat{X}_i$, then from (7.13) and (7.14) we have the following observer error dynamics

$$\dot{\tilde{X}}_i = -Q_i(\eta_i)\overline{M}_i^{-1}\overline{D}_i Q_i^{-1}(\eta_i)\tilde{X}_i - \Xi_i. \tag{7.21}$$

Considering the Lyapunov function candidate

$$V_{oi} = 0.5\|\tilde{X}_i\|^2 \tag{7.22}$$

whose derivative along the solutions of (7.21) satisfies

$$\dot{V}_{oi} = -\tilde{X}_i^T Q_i(\eta_i)\overline{M}_i^{-1}\overline{D}_i Q_i^{-1}(\eta_i)\tilde{X}_i - \tilde{X}_i^T \Xi_i \leq -\rho_{oi}\|\tilde{X}_i\|^2 - \tilde{X}_i^T \Xi_i \tag{7.23}$$

where the strictly positive constant ρ_{oi} is defined as $\rho_{oi} = \lambda_{min}(\overline{M}_i^{-1}\overline{D}_i)$ with $\lambda_{min}(\overline{M}_i^{-1}\overline{D}_i)$ the minimum eigenvalue of $(\overline{M}_i^{-1}\overline{D}_i)$. We define an estimate of ϖ_i as

$$\hat{\varpi}_i = Q_i^{-1}(\eta_i)\hat{X}_i \tag{7.24}$$

then we have

$$\tilde{\varpi}_i = Q_i^{-1}(\eta_i)\tilde{X}_i. \tag{7.25}$$

We will use the equations (7.22), (7.23) and (7.25) in the proof of the main result. Now using (7.14) and (7.24), we can write the robot dynamics (7.1) for formation control design in the next section as:

$$\begin{aligned}
\dot{x}_i &= \hat{v}_i\cos(\phi_i) + \tilde{v}_i\cos(\phi_i)\\
\dot{y}_i &= \hat{v}_i\sin(\phi_i) + \tilde{v}_i\sin(\phi_i)\\
\dot{\phi}_i &= \hat{w}_i + \tilde{w}_i\\
\overline{M}_i\dot{\hat{\varpi}}_i &= -\overline{C}_i(\hat{w}_i)\hat{\varpi}_i - \overline{D}_i\hat{\varpi}_i + \overline{B}_i\tau_i - \overline{C}_i(\tilde{w}_i)\hat{\varpi}_i + \overline{M}_i Q_i^{-1}(\eta_i)\Xi_i
\end{aligned} \tag{7.26}$$

for all $i = 1, ..., N$ where $\hat{v}_i$, $\hat{w}_i$ and $\tilde{v}_i$, $\tilde{w}_i$ are defined from $\hat{\varpi}_i = [\hat{v}_i\ \hat{w}_i]^T$ and $\tilde{\varpi}_i = [\tilde{v}_i\ \tilde{w}_i]^T$.

7.3 Control design

Since the robot dynamics (7.26) is of a strict feedback form [12] with respect to the estimate of the robot linear velocity, $\hat{v}_i$, and the estimate of the angular velocity, $\hat{w}_i$, we will use the backstepping technique [12] to design the control input τ_i. The control design is divided into two main stages. At the first stage, we consider the first three equations of (7.26) with $\hat{v}_i$ and $\hat{w}_i$ being considered as immediate controls. At the second stage, the actual control τ_i will be designed.

7.3.1 Stage I

Since the robot is underactuated, we divide this stage into two steps using the backstepping technique. At the first step, the robot heading, ϕ_i, and the estimate of the robot linear velocity, $\hat{v}_i$, are used as immediate controls to fulfill the task of position tracking and collision avoidance. At the second step, the estimate of the robot angular velocity, $\hat{w}_i$, is used as an immediate control to stabilize the error between the actual robot heading and its immediate value at the origin. We do not use the transformation in [14] to interpret the tracking errors in a frame attached to the reference trajectories as often done in literature (e.g. [15], [1], [16], [18]) to avoid difficulties when dealing with collision avoidance.

Step I.1

Define

$$\phi_{ie} = \phi_i - \alpha_{\phi_i}, \;\; v_{ie} = \hat{v}_i - \alpha_{\hat{v}_i} \tag{7.27}$$

where α_{ϕ_i} and $\alpha_{\hat{v}_i}$ are virtual controls of ϕ_i and $\hat{v}_i$, respectively. With (7.27), the first two equations of (7.26) are read:

$$\dot{q}_i = u_i + \Delta_{1i} + \Delta_{2i} + \Delta_{3i} \tag{7.28}$$

where $q_i = [x_i \;\; y_i]^T$, and

$$\begin{aligned}
u_i &= \begin{bmatrix} \cos(\alpha_{\phi_i}) \\ \sin(\alpha_{\phi_i}) \end{bmatrix} \alpha_{\hat{v}_i}, \\
\Delta_{1i} &= \begin{bmatrix} (\cos(\phi_{ie}) - 1)\cos(\alpha_{\phi_i}) - \sin(\phi_{ie})\sin(\alpha_{\phi_i}) \\ (\sin(\phi_{ie})\cos(\alpha_{\phi_i}) + (\cos(\phi_{ie}) - 1)\sin(\alpha_{\phi_i}) \end{bmatrix} \alpha_{\hat{v}_i}, \\
\Delta_{2i} &= \begin{bmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{bmatrix} v_{ie}, \\
\Delta_{3i} &= \begin{bmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{bmatrix} \tilde{v}_i.
\end{aligned} \tag{7.29}$$

To fulfill the task of position tracking and collision avoidance, we consider the following potential function

$$\varphi_{J1} = \sum_{i=1}^{N} (\gamma_i + 0.5\beta_i) \tag{7.30}$$

where γ_i and β_i are the goal and related collision avoidance functions for the robot i specified as follows:

-The goal function is designed such that it puts penalty on the tracking error for the robot, and is equal to zero when the robot is at its desired position. A simple choice of this function is

$$\gamma_i = 0.5\|q_i - q_{id}\|^2. \tag{7.31}$$

-The related collision function β_i should be chosen such that it is equal to infinity whenever any robots come in contact with the robot i , i.e. a collision occurs, and attains the minimum value when the robot i is at its desired location with respect to other group members belong to the set $\mathbb{N}_i$ robots, where $\mathbb{N}_i$ is the set that contains all the robots in the group except for the robot i. This function is chosen as follows:

$$\beta_i = \sum_{j\in\mathbb{N}_i} \beta_{ij} \tag{7.32}$$

where the function $\beta_{ij} = \beta_{ji}$ is a function of $\|q_{ij}\|^2/2$ with $q_{ij} = q_i - q_j$. This function is chosen as follows:

$$\beta_{ij} = \frac{h_{ij}\left(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\right)}{1 - h_{ij}\left(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\right)} \tag{7.33}$$

where $h_{ij}\left(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\right)$ is a p times differentiable jump function defined in Definition A.28 with $p \geq 3$ and $a_{ij} \geq (\underline{R}_i + \underline{R}_j)$, and $b_{ij} \leq \min(\overline{R}_i, \overline{R}_j, \|q_{ijd}\|)$. It can be readily checked that the function β_{ij} given in (7.33) enjoys the following properties:

$$\begin{aligned}
&1)\quad \beta_{ij} = 0,\ \beta'_{ij} = 0,\ \beta''_{ij} \geq 0 \quad \text{if} \quad \|q_{ij}\| = \|q_{ijd}\|,\\
&2)\quad \beta_{ij} > 0 \quad \text{if} \quad 0 < \|q_{ij}\| < b_{ij},\\
&3)\quad \beta_{ij} = 0,\ \beta'_{ij} = 0,\ \beta''_{ij} = 0,\ \beta'''_{ij} = 0 \quad \text{if} \quad \|q_{ij}\| \geq b_{ij},\\
&4)\quad \beta_{ij} = \infty \quad \text{if} \quad \|q_{ij}\| \leq (\underline{R}_i + \underline{R}_j),\\
&5)\quad \beta_{ij} \leq \mu_1,\ |\beta'_{ij}| \leq \mu_2,\ \text{and}\ |\beta''_{ij} q_{ij}^T q_{ij}| \leq \mu_3,\ \forall (\underline{R}_i + \underline{R}_j) < \|q_{ij}\| \leq b_{ij},\\
&6)\quad \beta_{ij} \text{ is at least three times differentiable}\\
&\qquad \text{with respect to } \|q_{ij}\|^2/2 \text{ if } \|q_{ij}\| > (\underline{R}_i + \underline{R}_j)
\end{aligned} \tag{7.34}$$

where $q_{ijd} = q_{id} - q_{jd}$, b_{ij} is a strictly positive constant such that $(\underline{R}_i + \underline{R}_j) < b_{ij} \leq \min(\overline{R}_i, \overline{R}_j, \|q_{ijd}\|)$; μ_1, μ_2 and μ_3 are positive constants; $\beta'_{ij}, \beta''_{ij}$ and β'''_{ij} are defined as follows: $\beta'_{ij} = \infty, \beta''_{ij} = \infty, \beta'''_{ij} = \infty$ if $\|q_{ij}\| \leq (\underline{R}_i + \underline{R}_j)$; $\beta'_{ij} = \frac{\partial \beta_{ij}}{\partial(\|q_{ij}\|^2/2)}, \beta''_{ij} = \frac{\partial^2 \beta_{ij}}{\partial(\|q_{ij}\|^2/2)^2}, \beta'''_{ij} = \frac{\partial^3 \beta_{ij}}{\partial(\|q_{ij}\|^2/2)^3}$ if $\|q_{ij}\| > (\underline{R}_i + \underline{R}_j)$.

Remark 7.4. Properties 1) - 4) imply that the function β_i is positive definite, is equal to zero when all the robots are at their desired locations, and is equal to infinity when a collision between any robots in the group occurs. Moreover, Property 1) and the function γ_i given in (7.31) ensure that the function φ_{I1} attains the (unique) minimum value of zero when all the robots are at their desired positions. Also, Property 3) ensures that the collision avoidance between the robots i and j is only taken into account when they are in their communication areas. Property 5) is used to prove stability of the closed loop system. Property 6) ensures that we can use techniques such as the backstepping and direct Lyapunov design methods ([12], [58]) for control design and stability analysis for continuous systems instead of techniques for switched, nonsmooth and discontinuous systems ([59], [33]) to handle the collision avoidance problem under the robot limited sensing ranges.

The time derivative of φ_{I1} along the solutions of (7.28) satisfies

$$\dot{\varphi}_{I1} = \sum_{i=1}^{N} \Omega_i^T \left(u_i + \Delta_{1i} + \Delta_{2i} + \Delta_{3i} - \dot{q}_{od}\right) \tag{7.35}$$

where we have used $\dot{q}_{id} = \dot{q}_{od}, u_i - u_j = u_i - \dot{q}_{od} - (u_j - \dot{q}_{od}), \forall (i,j) \in (1,2,...,N), i \neq j$, and

$$\Omega_i = q_i - q_{id} + \sum_{j \in \mathbb{N}_i} \beta'_{ij} q_{ij}. \tag{7.36}$$

From (7.35), we choose a bounded control u_i designed as follows:

$$u_i = -k_0 u_{od}^2 \Psi(\Omega_i) + \dot{q}_{od} \tag{7.37}$$

where $\Psi(\Omega_i)$ denotes a vector of bounded functions of elements of Ω_i in the sense that $\Psi(\Omega_i) = \left[\psi(\Omega_{ix}) \ \psi(\Omega_{iy})\right]^T$ with Ω_{ix} and Ω_{iy} the first and second rows of Ω_i, i.e. $\Omega_i = [\Omega_{ix} \ \Omega_{iy}]^T$. The function $\psi(x)$ is a scalar, at least three times differentiable and bounded function with respect to x, and satisfies

$$\begin{aligned}
&1)\ |\psi(x)| \leq \varrho_1, \\
&2)\ \psi(x) = 0 \quad \text{if } x = 0, \quad x\psi(x) > 0 \text{ if } x \neq 0, \\
&3)\ \psi(-x) = -\psi(x), (x-y)[\psi(x) - \psi(y)] \geq 0, \\
&4)\ |\psi(x)/x| \leq \varrho_2, |\partial\psi(x)/\partial x| \leq \varrho_3, \partial\psi(x)/\partial x|_{x=0} = 1
\end{aligned} \tag{7.38}$$

for all $x \in \mathbb{R}, y \in \mathbb{R}$, where $\varrho_1, \varrho_2, \varrho_3$ are strictly positive constants. Some functions that satisfy the above properties are $\arctan(x)$ and $\tanh(x)$. The strictly positive constant k_0 is chosen such that

$$k_0 < \frac{1}{2\varrho_1 u_{od}^{max}}. \tag{7.39}$$

The above condition ensures that α_{ϕ_i} and $\alpha_{\dot{v}_i}$ are solvable from u_i. We now need to solve for α_{ϕ_i} and $\alpha_{\dot{v}_i}$ from the expression of u_i in (7.37) and (7.29). From (7.37) and (7.29), we have

$$\begin{aligned}\cos(\alpha_{\phi_i})\alpha_{\dot{v}_i} &= -k_0 u_{od}^2 \psi(\Omega_{ix}) + \cos(\phi_{od})u_{od},\\ \sin(\alpha_{\phi_i})\alpha_{\dot{v}_i} &= -k_0 u_{od}^2 \psi(\Omega_{iy}) + \sin(\phi_{od})u_{od}\end{aligned} \tag{7.40}$$

where we have used $\dot{x}_{od} = x'_{od}\dot{s}_{od} = \frac{x'_{od}\sqrt{x'^2_{od}+y'^2_{od}}\,\dot{s}_{od}}{\sqrt{x'^2_{od}+y'^2_{od}}} = \cos(\phi_{od})u_{od}$ and $\dot{y}_{od} = y'_{od}\dot{s}_{od} = \frac{y'_{od}\sqrt{x'^2_{od}+y'^2_{od}}\,\dot{s}_{od}}{\sqrt{x'^2_{od}+y'^2_{od}}} = \sin(\phi_{od})u_{od}$ since $\phi_{od} = \arctan(y'_{od}/x'_{od})$ and $\sqrt{x'^2_{od}+y'^2_{od}} > 0$, see Assumption 7.1. The left hand sides of (7.40) are actually the coordinates of u_i in the x and y directions. Now multiplying both sides of the first equation of (7.40) with $\cos(\phi_{od})$ and both sides of the second equation of (7.40) with $\sin(\phi_{od})$ then adding the resulting equations together yield

$$\cos(\alpha_{\phi_i}-\phi_{od})\alpha_{\dot{v}_i} = -k_0 u_{od}^2\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + u_{od}. \tag{7.41}$$

On the other hand multiplying both sides of the first equation of (7.40) with $\sin(\phi_{od})$ and both sides of the second equation of (7.40) with $\cos(\phi_{od})$ then subtracting the resulting equations give

$$\sin(\alpha_{\phi_i}-\phi_{od})\alpha_{\dot{v}_i} = -k_0 u_{od}^2\big(-\psi(\Omega_{ix})\sin(\phi_{od}) + \psi(\Omega_{ix})\cos(\phi_{od})\big). \tag{7.42}$$

From (7.41) and (7.42), we solve for α_{ϕ_i} and $\alpha_{\dot{v}_i}$ as

$$\begin{aligned}\alpha_{\phi_i} &= \phi_{od} + \arctan\left(\frac{-k_0 u_{od}\big(-\psi(\Omega_{ix})\sin(\phi_{od}) + \psi(\Omega_{ix})\cos(\phi_{od})\big)}{-k_0 u_{od}\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + 1}\right),\\ \alpha_{\dot{v}_i} &= \cos(\alpha_{\phi_i})\big(-k_0 u_{od}^2\psi(\Omega_{ix}) +\\ &\qquad \cos(\phi_{od})u_{od}\big) + \sin(\alpha_{\phi_i})\big(-k_0 u_{od}^2\psi(\Omega_{iy}) + \sin(\phi_{od})u_{od}\big).\end{aligned} \tag{7.43}$$

It is noted that (7.43) is well defined since

$$-k_0 u_{od}\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + 1 \geq -2\varrho_1 k_0 u_{od}^{max} + 1 > 0$$

where the condition (7.39) has been used. Moreover, it is of interest to note that α_{ϕ_i} and $\alpha_{\dot{v}_i}$ are at least twice differentiable functions of $q_{od}, \phi_{od}, u_{od}, q_i, q_{ij}$ with $j \in \mathbb{N}_i, j \neq i$.

Remark 7.5. When Ω_i defined in (7.36) is substituted into (7.37), the control u_i can be written as

$$u_i = k_0 u_{od}^2 \begin{bmatrix} \psi\big(-(x_i - x_{id}) - \sum_{j\in\mathbb{N}_i}\beta'_{ij}(x_i - x_j)\big) \\ \psi\big(-(y_i - y_{id}) - \sum_{j\in\mathbb{N}_i}\beta'_{ij}(y_i - y_j)\big)\end{bmatrix} + \dot{q}_{od}. \tag{7.44}$$

It is seen from (7.44) that the argument of ψ consists of two parts. The first part, $-(x_i - x_{id})$ or $-(y_i - y_{id})$, referred to as the attractive force plays the role of forcing the robot i to its desired location. The second part, $-\sum_{j\in\mathbb{N}_i}\beta'_{ij}(x_i-x_j)$ or $-\sum_{j\in\mathbb{N}_i}\beta'_{ij}(y_i-y_j)$, referred to as the repulsive force, takes care of collision avoidance for the robot i with the other robots. Moreover, the immediate control u_i of the robot i given in (7.37) depends on only its own state and reference trajectory, and the states of other neighbor robots j if the points P_{0j} of these robots are in the circular communication area of the robot i, since outside this area $\beta'_{ij} = 0$, see Property 3) of β_{ij}.

Now substituting (7.37) into (7.35) results in

$$\dot{\varphi}_{I1} = -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) + \sum_{i=1}^{N} \Omega_i^T (\Delta_{1i} + \Delta_{2i} + \Delta_{3i}). \tag{7.45}$$

Substituting (7.37) into (7.28) results in

$$\dot{q}_i = -k_0 u_{od}^2 \Psi(\Omega_i) + \dot{q}_{od} + \Delta_{1i} + \Delta_{2i} + \Delta_{3i}. \tag{7.46}$$

Step I.2

At this step, we view $\hat{w}_i$ as an immediate control to stabilize ϕ_{ie} at the origin. As such, we define

$$w_{ie} = \hat{w}_i - \alpha_{\hat{w}_i} \tag{7.47}$$

where $\alpha_{\hat{w}_i}$ is a virtual control of $\hat{w}_i$. To prepare for design of the virtual control $\alpha_{\hat{w}_i}$, let us calculate $\dot{\phi}_{ie}$. Differentiating both sides of the first equation of (7.27) along the solutions of (7.47), the third equation of (7.26), and the second equation of (7.43) results in

$$\begin{aligned}\dot{\phi}_{ie} =& w_{ie} + \alpha_{\hat{w}_i} - \frac{\partial \alpha_{\phi_i}}{\partial q_{od}}\dot{q}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial \phi_{od}}\dot{\phi}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial u_{od}}\dot{u}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial q_i}(u_i + \Delta_{1i} + \\ & \Delta_{2i} + \Delta_{3i}) - \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}(u_i + \Delta_{1i} + \Delta_{2i} + \Delta_{3i} - (u_j + \Delta_{1j} + \\ & \Delta_{2j} + \Delta_{3j})) + \bar{w}_i.\end{aligned} \tag{7.48}$$

To design the virtual control $\alpha_{\hat{w}_i}$, we consider the following function

$$\varphi_{I2} = \varphi_{I1} + 0.5\sum_{i=1}^{N}\phi_{ie}^2 \tag{7.49}$$

whose derivative along the solutions of (7.45) and (7.48) satisfies

$$\dot{\varphi}_{I2} = -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) + \sum_{i=1}^{N} \Omega_i^T (\Delta_{2i} + \Delta_{3i}) + \sum_{i=1}^{N} \phi_{ie} \bigg(\frac{\Omega_i^T \Delta_{1i}}{\phi_{ie}} + w_{ie} + \alpha_{\hat{w}_i} + \tilde{w}_i - \frac{\partial \alpha_{\phi_i}}{\partial q_{od}} \dot{q}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial \phi_{od}} \dot{\phi}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial u_{od}} \dot{u}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial q_i} (u_i + \Delta_{1i} + \Delta_{2i} + \Delta_{3i}) - \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (u_i + \Delta_{1i} + \Delta_{2i} + \Delta_{3i} - (u_j + \Delta_{1j} + \Delta_{2j} + \Delta_{3j})) \bigg). \tag{7.50}$$

It is noted that $\frac{\Delta_{1i}}{\phi_{ie}}$ is well defined since $\frac{\sin(\phi_{ie})}{\phi_{ie}} = \int_0^1 \cos(\lambda \phi_{ie}) d\lambda$ and $\frac{\cos(\phi_{ie})-1}{\phi_{ie}} = \int_1^0 \sin(\lambda \phi_{ie}) d\lambda$ are smooth functions for all $\phi_{ie} \in \mathbb{R}$. From (7.50), we choose the virtual control $\alpha_{\hat{w}_i}$ as

$$\alpha_{\hat{w}_i} = -k_i \phi_{ie} - \frac{\Omega_i^T \Delta_{1i}}{\phi_{ie}} + \frac{\partial \alpha_{\phi_i}}{\partial q_{od}} \dot{q}_{od} + \frac{\partial \alpha_{\phi_i}}{\partial \phi_{od}} \dot{\phi}_{od} + \frac{\partial \alpha_{\phi_i}}{\partial u_{od}} \dot{u}_{od} + \frac{\partial \alpha_{\phi_i}}{\partial q_i} (u_i + \Delta_{1i}) + \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (u_i + \Delta_{1i} - (u_j + \Delta_{1j})) \tag{7.51}$$

where k_i is a positive constant.

Remark 7.6. The virtual control $\alpha_{\hat{w}_i}$ is at least once differentiable function of $q_{od}, \dot{q}_{od}, \phi_{od}, \dot{\phi}_{od}, u_{od}, \dot{u}_{od}, q_i, \phi_i, q_{ij}, \phi_j$ with $j \in \mathbb{N}_i, j \neq i$, and contains only the state and reference trajectory of the robot i, and the states of other neighbor robots j if the points P_{0j} of these robots are in the communication area of the robot i, because outside this area $\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} = 0$ thanks to Property 3) of β_{ij}. Moreover, we did not include any nonlinear damping terms in the virtual control $\alpha_{\hat{w}_i}$ to handle the terms such as $\phi_{ie}\tilde{w}_i$, $\frac{\partial \alpha_{\phi_i}}{\partial q_i} \Delta_{3i}, \ldots$ involving with the observer errors $\tilde{w}_i$ and $\tilde{v}_i$ in (7.50). These terms will be cancelled when we design the interlaced term Ξ_i at the final stage.

Substituting (7.51) into (7.50) results in (after some manipulation):

$$\dot{\varphi}_{I2} = -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) - \sum_{i=1}^{N} k_i \phi_{ie}^2 + \sum_{i=1}^{N} \phi_{ie} \bigg(-\frac{\partial \alpha_{\phi_i}}{\partial q_i} \Delta_{3i} - \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (\Delta_{3i} - \Delta_{3j}) \bigg) + \sum_{i=1}^{N} \bigg[\phi_{ie} (w_{ie} + \tilde{w}_i) + \bigg(\Omega_i^T - \phi_{ie} \frac{\partial \alpha_{\phi_i}}{\partial q_i} - \sum_{j=1, j\neq i}^{N} \Big(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} \phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}} \phi_{je} \Big) \bigg) \Delta_{2i} \bigg]. \tag{7.52}$$

Substituting (7.51) into (7.48) gives

$$\begin{aligned}\dot{\phi}_{ie} =& -k_i\phi_{ie} - \frac{\Omega_i^T\Delta_{1i}}{\phi_{ie}} - \frac{\partial\alpha_{\phi_i}}{\partial q_i}(\Delta_{2i}+\Delta_{3i}) - \\ & \sum_{j=1,j\neq i}^{N}\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}\Big(\Delta_{2i}+\Delta_{3i}-(\Delta_{2j}+\Delta_{3j})\Big) + w_{ie} + \tilde{w}_i. \end{aligned} \tag{7.53}$$

To prepare for the next section, let us compute the term $\overline{M}_i\dot{\varpi}_{ie}$ where $\varpi_{ie} = [v_{ie}\ w_{ie}]^T$. From the second equation of (7.27), (7.47), and the last equation of (7.26), we have

$$\begin{aligned}\overline{M}_i\dot{\varpi}_{ie} =& -\overline{C}_i(\hat{w}_f)\hat{\varpi}_i - \overline{D}_i\hat{\varpi}_i - \overline{M}_i[\dot{\alpha}_{\hat{v}_i}\ \dot{\alpha}_{\hat{w}_i}]^T + \\ & \overline{B}_i\tau_i - \overline{C}_i(\tilde{w}_i)\hat{\varpi}_i + \overline{M}_iQ_i^{-1}(\eta_i)\Xi_i \end{aligned} \tag{7.54}$$

where

$$\begin{aligned}\dot{\alpha}_{\hat{v}_i} =& \bar{\dot{\alpha}}_{\hat{v}_i} + \bar{\bar{\dot{\alpha}}}_{\hat{v}_i}, \\ \bar{\dot{\alpha}}_{\hat{v}_i} =& \frac{\partial\alpha_{\hat{v}_i}}{\partial q_{od}}\dot{q}_{od} + \frac{\partial\alpha_{\hat{v}_i}}{\partial u_{od}}\dot{u}_{od} + \frac{\partial\alpha_{\hat{v}_i}}{\partial\phi_{od}}\dot{\phi}_{od} + \frac{\partial\alpha_{\hat{v}_i}}{\partial q_i}\vartheta_i + \sum_{j=1,j\neq i}^{N}\frac{\partial\alpha_{\hat{v}_i}}{\partial q_{ij}}(\vartheta_i-\vartheta_j), \\ \bar{\bar{\dot{\alpha}}}_{\hat{v}_i} =& \frac{\partial\alpha_{\hat{v}_i}}{\partial q_i}\Delta_{3i} + \sum_{j=1,j\neq i}^{N}\frac{\partial\alpha_{\hat{v}_i}}{\partial q_{ij}}(\Delta_{3i}-\Delta_{3j}),\end{aligned} \tag{7.55}$$

and

$$\begin{aligned}\dot{\alpha}_{\hat{w}_i} =& \bar{\dot{\alpha}}_{\hat{w}_i} + \bar{\bar{\dot{\alpha}}}_{\hat{w}_i}, \\ \bar{\dot{\alpha}}_{\hat{w}_i} =& \frac{\partial\alpha_{\hat{w}_i}}{\partial q_{od}}\dot{q}_{od} + \frac{\partial\alpha_{\hat{w}_i}}{\partial\dot{q}_{od}}\ddot{q}_{od} + \frac{\partial\alpha_{\hat{w}_i}}{\partial u_{od}}\dot{u}_{od} + \frac{\partial\alpha_{\hat{w}_i}}{\partial\dot{u}_{od}}\ddot{u}_{od} + \frac{\partial\alpha_{\hat{w}_i}}{\partial\phi_{od}}\dot{\phi}_{od} + \frac{\partial\alpha_{\hat{w}_i}}{\partial\dot{\phi}_{od}}\ddot{\phi}_{od} + \\ & \frac{\partial\alpha_{\hat{w}_i}}{\partial q_i}\vartheta_i + \frac{\partial\alpha_{\hat{w}_i}}{\partial\phi_i}\hat{w}_i + \sum_{j=1,j\neq i}^{N}\left(\frac{\partial\alpha_{\hat{w}_i}}{\partial\phi_j}\hat{w}_j + \frac{\partial\alpha_{\hat{w}_i}}{\partial q_{ij}}(\vartheta_i-\vartheta_j)\right), \\ \bar{\bar{\dot{\alpha}}}_{\hat{w}_i} =& \frac{\partial\alpha_{\hat{w}_i}}{\partial q_i}\Delta_{3i} + \frac{\partial\alpha_{\hat{w}_i}}{\partial\phi_i}\tilde{w}_i + \sum_{j=1,j\neq i}^{N}\left(\frac{\partial\alpha_{\hat{w}_i}}{\partial\phi_j}\tilde{w}_j + \frac{\partial\alpha_{\hat{w}_i}}{\partial q_{ij}}(\Delta_{3i}-\Delta_{3j})\right)\end{aligned} \tag{7.56}$$

with $\vartheta_i = u_i + \Delta_{1i} + \Delta_{2i}, i = 1, ..., N$. Again, $\dot{\alpha}_{\hat{v}_i}$ and $\dot{\alpha}_{\hat{w}_i}$ contain only the state and reference trajectory of the robot i, and the states of other neighbor robots j if the points P_{0j} of these robots are in the communication area of the robot i, because outside this area $\frac{\partial\alpha_{\hat{v}_i}}{\partial q_{ij}} = 0$, $\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}} = 0$, and $\frac{\partial\alpha_{\hat{w}_i}}{\partial\phi_j} = 0$ thanks to Property 3) of β_{ij}.

7.3.2 Stage II

At this stage, we design the actual control input vector τ_i for each robot i. To do so, we consider the following function

$$\varphi_{II} = \varphi_{I2} + \frac{1}{2}\sum_{i=1}^{N} \varpi_{ie}^T \overline{M}_i \varpi_{ie}. \tag{7.57}$$

Differentiating both sides of (7.57) along the solutions of (7.54) and (7.52) yields

$$\begin{aligned}\dot{\varphi}_{II} &= -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) - \sum_{i=1}^{N} k_i \phi_{ie}^2 + \sum_{i=1}^{N} \phi_{ie} w_{ie} \\ &+ \sum_{i=1}^{N} \left(\Omega_i^T - \phi_{ie}\frac{\partial \alpha_{\phi_i}}{\partial q_i} - \sum_{j=1, j\neq i}^{N} \left(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\right)\right)\Delta_{2i} + \sum_{i=1}^{N} \varpi_{ie}^T \times \\ &\left(- \overline{C}_i(\hat{w}_i)\hat{\varpi}_i - \overline{D}_i\hat{\varpi}_i - \overline{M}_i[\bar{\dot{\alpha}}_{\hat{v}_i}\ \bar{\dot{\alpha}}_{\hat{w}_i}]^T + \overline{B}_i\tau_i + \overline{M}_i Q_i^{-1}(\eta_i)\Xi_i\right) + \Lambda\end{aligned} \tag{7.58}$$

where $\bar{\dot{\alpha}}_{\hat{v}_i}$ and $\bar{\dot{\alpha}}_{\hat{w}_i}$ are given in (7.55) and (7.56), respectively, and

$$\begin{aligned}\Lambda = \sum_{i=1}^{N} \Bigg(&\phi_{ie}\tilde{w}_i + \Omega_i^T \Delta_{3i} - \phi_{ie}\left(\frac{\partial \alpha_{\phi_i}}{\partial q_i}\Delta_{3i} + \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}}(\Delta_{3i} - \Delta_{3j})\right) - \\ &\varpi_{ie}^T \overline{M}_i \begin{bmatrix} \frac{\partial \alpha_{\hat{v}_i}}{\partial q_i}\Delta_{3i} + \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\hat{v}_i}}{\partial q_{ij}}(\Delta_{3i} - \Delta_{3j}) \\ \frac{\partial \alpha_{\hat{w}_i}}{\partial q_i}\Delta_{3i} + \frac{\partial \alpha_{\hat{w}_i}}{\partial \phi_i}\tilde{w}_i + \sum_{j=1, j\neq i}^{N}\left(\frac{\partial \alpha_{\hat{w}_i}}{\partial \phi_j}\tilde{w}_j + \frac{\partial \alpha_{\hat{w}_i}}{\partial q_{ij}}(\Delta_{3i} - \Delta_{3j})\right)\end{bmatrix} - \\ &\varpi_{ie}^T \overline{M}_i \overline{C}_i(\tilde{w}_i)\hat{\varpi}_i \Bigg)\end{aligned} \tag{7.59}$$

and we have used (7.55) and (7.56). After some rearrangement, we can write Λ as

$$\Lambda = \sum_{i=1}^{N} [\overline{\Lambda}_{1i}\ \overline{\Lambda}_{2i}]^T \tilde{\varpi}_i = \sum_{i=1}^{N} [\overline{\Lambda}_{1i}\ \overline{\Lambda}_{2i}]^T Q_i^{-1}(\eta_i)\tilde{X}_i \tag{7.60}$$

where we have used (7.25), and $\overline{\Lambda}_{1i}$ and $\overline{\Lambda}_{2i}$ are given by

$$\overline{\Lambda}_{1i} = \Big(\Omega_i^T - \phi_{ie}\frac{\partial\alpha_{\phi_i}}{\partial q_i} - \sum_{j=1,j\neq i}^{N}\Big[\phi_{ie}\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}} - \phi_{je}\frac{\partial\alpha_{\phi_j}}{\partial q_{ij}}\Big) -$$
$$A_1\Big(\frac{\partial\alpha_{\hat{v}_i}}{\partial q_i} + \sum_{j=1,j\neq i}^{N}\Big(\frac{\partial\alpha_{\hat{v}_i}}{\partial q_{ij}} - \frac{\partial\alpha_{\hat{v}_j}}{\partial q_{ij}}\Big)\Big) -$$
$$A_2\Big(\frac{\partial\alpha_{\hat{w}_i}}{\partial q_i} + \sum_{j=1,j\neq i}^{N}\Big(\frac{\partial\alpha_{\hat{w}_i}}{\partial q_{ij}} - \frac{\partial\alpha_{\hat{w}_j}}{\partial q_{ij}}\Big)\Big)\Big]\bar{\Delta}_{3i}$$
$$\overline{\Lambda}_{2i} = \phi_{ie} - \varpi_{ie}^T\overline{M}_i\overline{C}_i(1)\hat{\varpi}_i - A_2\frac{\partial\alpha_{\hat{w}_i}}{\partial\phi_i} - A_2\sum_{j=1,j\neq i}^{N}\frac{\partial\alpha_{\hat{w}_j}}{\partial\phi_i} \tag{7.61}$$

where $\overline{C}_i(1)$ is $\overline{C}_i(\tilde{w}_i)$ with $\tilde{w}_i$ replaced by 1, $\bar{\Delta}_{3i} = [\cos(\phi_i)\ \ \sin(\phi_i)]^T$, A_1 and A_2 are defined by $[A_1\ A_2] = \varpi_{ie}^T\overline{M}_i$. Now, we take the following total Lyapunov function candidate

$$\varphi_{TOT} = \varphi_{II} + \sum_{i=1}^{N} V_{oi} \tag{7.62}$$

whose derivative along the solutions of (7.58) and (7.23) is

$$\dot{\varphi}_{TOT} \leq -k_0 u_{od}^2\sum_{i=1}^{N}\Omega_i^T\Psi(\Omega_i) - \sum_{i=1}^{N}k_i\phi_{ie}^2 - \sum_{i=1}^{N}\rho_{oi}\|\tilde{X}_i\|^2 +$$
$$\sum_{i=1}^{N}\phi_{ie}w_{ie} + \sum_{i=1}^{N}\Big(\Omega_i^T - \phi_{ie}\frac{\partial\alpha_{\phi_i}}{\partial q_i} - \sum_{j=1,j\neq i}^{N}\Big(\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial\alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\Big)\Big)\Delta_{2i} +$$
$$\sum_{i=1}^{N}\varpi_{ie}^T\Big(-\overline{C}_i(\hat{w}_i)\hat{\varpi}_i - \overline{D}_i\hat{\varpi}_i - \overline{M}_i[\dot{\bar{\alpha}}_{\hat{v}_i}\ \dot{\bar{\alpha}}_{\hat{w}_i}]^T + \overline{B}_i\tau_i + \overline{M}_iQ_i^{-1}(\eta_i)\Xi_i\Big) +$$
$$\sum_{i=1}^{N}\Big([\overline{\Lambda}_{1i}\ \overline{\Lambda}_{2i}]^TQ_i^{-1}(\eta_i) - \Xi_i^T\Big)\tilde{X}_i. \tag{7.63}$$

The inequality (7.63) suggests that we design the control τ_i and the interlaced term Ξ_i as

$$\tau_i = \overline{B}_i^{-1}\Big(-L_i\varpi_{ie} + \overline{C}_i(\hat{w}_i)\hat{\varpi}_i + \overline{D}_i[\alpha_{\hat{v}_i}\ \alpha_{\hat{w}_i}]^T + \overline{M}_i[\dot{\bar{\alpha}}_{\hat{v}_i}\ \dot{\bar{\alpha}}_{\hat{w}_i}]^T -$$
$$\overline{M}_iQ_i^{-1}(\eta_i)\Xi_i -$$
$$\begin{bmatrix}\Big(\Omega_i^T - \phi_{ie}\frac{\partial\alpha_{\phi_i}}{\partial q_i} - \sum_{j=1,j\neq i}^{N}\Big(\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial\alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\Big)\Big)\bar{\Delta}_{2i}\\ \phi_{ie}\end{bmatrix}\Big)$$
$$\Xi_i = \Big([\overline{\Lambda}_{1i}\ \overline{\Lambda}_{2i}]^TQ_i^{-1}(\eta_i)\Big)^T \tag{7.64}$$

where $\bar{\Delta}_{2i} = [\cos(\phi_i) \;\; \sin(\phi_i)]^T$, and L_i a symmetric positive definite matrix. Substituting (7.64) into (7.54) gives

$$\overline{M}_i\dot{\varpi}_{ie} = -(\overline{D}_i + L_i)\varpi_{ie} - \overline{M}_i[\bar{\bar{\dot{\alpha}}}_{\hat{v}_i} \;\; \bar{\bar{\dot{\alpha}}}_{\hat{w}_i}]^T - \overline{C}_i(\tilde{w}_i)\hat{\varpi}_i - \begin{bmatrix} \left(\Omega_i^T - \phi_{ie}\frac{\partial\alpha_{\phi_i}}{\partial q_i} - \sum_{j=1,j\neq i}^N\left(\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial\alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\right)\right)\bar{\Delta}_{2i} \\ \phi_{ie} \end{bmatrix} \quad (7.65)$$

where $\bar{\bar{\dot{\alpha}}}_{\hat{v}_i}$ and $\bar{\bar{\dot{\alpha}}}_{\hat{w}_i}$ are given in the last equation of (7.55) and (7.56), respectively. By construction, the control τ_i and the interlaced term Ξ_i given in (7.64) of the robot i contain only the state and reference trajectory of the robot i, and the states of other neighbor robots j if these robots are in a circular area, which is centered at point P_{0i} of the robot i and has a radius no greater than $\overline{R}_i$. Now substituting (7.64) into (7.63) results in

$$\dot{\varphi}_{TOT} \leq -k_0u_{od}^2\sum_{i=1}^N\Omega_i^T\Psi(\Omega_i) - \sum_{i=1}^N k_i\phi_{ie}^2 - \sum_{i=1}^N\varpi_{ie}^T(\overline{D}_i + L_i)\varpi_{ie} - \sum_{i=1}^N\rho_{oi}\|\tilde{X}_i\|^2. \quad (7.66)$$

For convenience, we rewrite the closed loop system consisting of (7.46), (7.53), (7.65), and (7.21) as follows:

$$\begin{aligned}
\dot{q}_i &= -k_0u_{od}^2\Psi(\Omega_i) + \dot{q}_{od} + \Delta_{1i} + \Delta_{2i} + \Delta_{3i}, \\
\dot{\phi}_{ie} &= -k_i\phi_{ie} - \frac{\Omega_i^T\Delta_{1i}}{\phi_{ie}} - \frac{\partial\alpha_{\phi_i}}{\partial q_i}(\Delta_{2i} + \Delta_{3i}) - \\
&\quad \sum_{j=1,j\neq i}^N\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}\Big(\Delta_{2i} + \Delta_{3i} - (\Delta_{2j} + \Delta_{3j})\Big) + w_{ie} + \tilde{w}_i, \\
\overline{M}_i\dot{\varpi}_{ie} &= -(\overline{D}_i + L_i)\varpi_{ie} - \overline{M}_i[\bar{\bar{\dot{\alpha}}}_{\hat{v}_i} \;\; \bar{\bar{\dot{\alpha}}}_{\hat{w}_i}]^T - \overline{C}_i(\tilde{w}_i)\hat{\varpi}_i - \\
&\quad \begin{bmatrix} \left(\Omega_i^T - \phi_{ie}\frac{\partial\alpha_{\phi_i}}{\partial q_i} - \sum_{j=1,j\neq i}^N\left(\frac{\partial\alpha_{\phi_i}}{\partial q_{ij}}\phi_{ie} - \frac{\partial\alpha_{\phi_j}}{\partial q_{ji}}\phi_{je}\right)\right)\bar{\Delta}_{2i} \\ \phi_{ie} \end{bmatrix}, \\
\dot{\tilde{X}}_i &= -Q_i(\eta_i)\overline{M}_i^{-1}\overline{D}_iQ_i^{-1}(\eta_i)\tilde{X}_i - \Xi_i.
\end{aligned} \quad (7.67)$$

We now state the main result of this chapter in the following theorem.

Theorem 7.7. *Under Assumption 7.1, the control τ_i and the observer $\dot{\hat{X}}_i$ given in (7.64) and (7.14) for the robot i solve the formation control objective. In particular, no collisions between any robots can occur for all $t \geq t_0 \geq 0$, the closed loop system (7.67) is forward complete, and the position and orientation of the robots track their reference trajectories asymptotically in the sense of (7.6).*

7.4 Proof of Theorem 7.7

We first prove that no collisions between the robots can occur, the closed loop system (7.67) is forward complete, and that the robots asymptotically approach their target points or some critical points. We then investigate stability of the closed loop system (7.67) at these points, and show that the position and orientation of the robots asymptotically track their reference trajectories.

+Proof of no collisions and complete forwardness of closed loop system. From (7.66) and properties of the function ψ , see (7.38), we have $\dot{\varphi}_{TOT} \leq 0 \leq 0$, which implies that $\varphi_{TOT}(t) \leq \varphi_{TOT}(t_0), \forall t \geq t_0$. With definition of the function φ_{TOT} in (7.62) and its components in (7.49), (7.30), (7.31), (7.32), and (7.22), we have

$$\sum_{i=1}^{N}\Big[\gamma_i(t)+\frac{1}{2}\sum_{j\in\mathbb{N}_i}\beta_{ij}(t)+\frac{1}{2}\phi_{ie}^2(t)+\frac{1}{2}\varpi_{ie}^T(t)\overline{M}_i\varpi_{ie}(t)+\frac{1}{2}\tilde{X}_i^T(t)\tilde{X}_i(t)\Big] \leq$$
$$\sum_{i=1}^{N}\Big[\gamma_i(t_0)+\frac{1}{2}\sum_{j\in\mathbb{N}_i}\beta_{ij}(t_0)+\frac{1}{2}\phi_{ie}^2(t_0)+$$
$$\frac{1}{2}\varpi_{ie}^T(t_0)\overline{M}_i\varpi_{ie}(t_0)+\frac{1}{2}\tilde{X}_i^T(t_0)\tilde{X}_i(t_0)\Big] \tag{7.68}$$

for all $t \geq t_0 \geq 0$. From the condition specified in item 4) of Assumption 7.1, and Property 5) of β_{ij}, and definition of ϕ_{ie}, ϖ_{ie}, we have the right hand side of (7.68) is bounded by a positive constant depending on the initial conditions. Boundedness of the right hand side of (7.68) implies that the left hand side of (7.68) must be also bounded. As a result, $\beta_{ij}(t)$ must be smaller than some positive constant depending on the initial conditions for all $t \geq t_0 \geq 0$. From properties of β_{ij}, see (7.34), $\|q_{ij}(t)\| - (\underline{R}_i + \underline{R}_j)$ must be larger than some positive constant depending on the initial conditions denoted by ϵ_3, i.e. there are no collisions for all $t \geq t_0 \geq 0$. Boundedness of the left hand side of (7.68) also implies that of $(q_i(t) - q_{id}(t)), \phi_{ie}(t), \varpi_{ie}(t)$ and $\tilde{X}_i(t)$ for all $t \geq t_0 \geq 0$. Therefore, the closed loop system (7.67) is forward complete.

+Equilibrium points. Since we have already proved that there are no collisions between any robots, an application of Theorem 8.4 in [58] to (7.66) yields

$$\lim_{t\to\infty}\Big(k_0u_{od}^2(t)\sum_{i=1}^{N}\Omega_i^T(t)\Psi(\Omega_i(t))+\sum_{i=1}^{N}k_i\phi_{ie}^2(t)+$$
$$\sum_{i=1}^{N}\varpi_{ie}^T(t)(\overline{D}_i+L_i)\varpi_{ie}(t)+\sum_{i=1}^{N}\rho_{oi}\|\tilde{X}_i(t)\|^2\Big)=0. \tag{7.69}$$

By noting that $\lim_{t\to\infty} u_{od}^2(t) \neq 0$ as specified in item 5) of Assumption 7.1, the limit equation (7.69) implies that

$$\lim_{t\to\infty} \Omega_i(t) = 0, \ \lim_{t\to\infty} \phi_{ie}(t) = 0, \ \lim_{t\to\infty} \varpi_{ie}(t) = 0, \ \lim_{t\to\infty} \tilde{X}_i(t) = 0.$$

It is of interest to note that $\lim_{t\to\infty} \tilde{X}_i(t) = 0$ implies that $\lim_{t\to\infty} \tilde{\varpi}_i(t) = 0$. This implies that the estimate of the i^{th} robot linear, $\hat{v}_i$, and angular, $\hat{w}_i$, velocities tend to the actual ones $v_i(t)$ and $w_i(t)$. By construction, $\lim_{t\to\infty} \Omega_i(t) = 0$ and $\lim_{t\to\infty} \phi_{ie}(t) = 0$ imply that $\lim_{t\to\infty}(\phi_i(t) - \phi_{od}(t)) = 0$. Moreover, from definition of Ω_i in (7.36), $\lim_{t\to\infty} \Omega_i(t) = 0$ means

$$\lim_{t\to\infty} \Big(q_i(t) - q_{id}(t) + \sum_{j\in\mathbb{N}_i} \beta'_{ij} q_{ij}(t) \Big) = 0. \tag{7.70}$$

The limit equation (7.70) implies that $q(t) = [q_1^T(t)\ q_2^T(t), ..., q_N^T(t)]^T$ can tend to $q_d = [q_{1d}^T\ q_{2d}^T, ..., q_{Nd}^T]^T$ since $\beta'_{ij}|_{\|q_{ij}\|=\|q_{ijd}\|} = 0$ (Property 1) of β_{ij}), or some vector denoted by $q_c = [q_{1c}^T\ q_{2c}^T, ..., q_{Nc}^T]^T$ as the time goes to infinity, i.e. the equilibrium points can be q_d or q_c. It is noted that some elements of q_c can be equal to that of q_d. However, for simplicity we abuse the notation, i.e. we still denote that vector as q_c. Indeed, the vector q_c is such that

$$\Omega_i|_{q=q_c} = \Big[q_i - q_{id} + \sum_{j\in\mathbb{N}_i} \beta'_{ij} q_{ij}\Big]\Big|_{q=q_c} = 0, \forall i = 1, ..., N. \tag{7.71}$$

To investigate properties of the equilibrium points, q_d and q_c, we consider the first equation of the closed loop system (7.67), i.e.

$$\dot{q}_i = -k_0 u_{od}^2 \Psi(\Omega_i) + \dot{q}_{od} + \Delta_{1i} + \Delta_{2i} + \Delta_{3i}. \tag{7.72}$$

Since we have already proved that the closed loop system (7.67) is forward complete, and $\lim_{t\to\infty} \phi_{ie}(t) = 0$, $\lim_{t\to\infty} \varpi_{ie}(t) = 0$, and $\lim_{t\to\infty} \tilde{X}_i(t) = 0$ imply from the expressions of Δ_{1i}, Δ_{2i} and Δ_{3i}, see (7.29) and (7.25), that $\lim_{t\to\infty}(\Delta_{1i}(t)+\Delta_{2i}(t)+\Delta_{3i}(t)) = 0$, we treat $\Delta_i(t) = \Delta_{1i}(t)+\Delta_{2i}(t)+\Delta_{3i}(t)$ as an input to (7.72) instead of a state. Moreover, we have already proved that the trajectory, q, can approach either the set of desired equilibrium points denoted by q_d or the set of undesired equilibrium points denoted by q_c 'almost globally'. The term 'almost globally' refers to the fact that the agents start from a set that includes both condition (7.2) and that does not coincide at any point with the set of the undesired saddle point q_c. Therefore, we now need to prove that q_d is locally asymptotically stable and that q_c is locally unstable. Once this is proved, we can conclude that the trajectory q approaches q_d from almost everywhere except for from the set denoted by the condition (7.2) and the set denoted by q_c, which is unstable.

+Properties of equilibrium points. The system (7.72) can be written in a vector form as

$$\dot{q} = -k_0 u_{od}^2 \Psi_q(q, q_d) + \text{vec}(\dot{q}_{od}) + \text{vec}(\Delta_i) \tag{7.73}$$

where $\Psi_q(q, q_d) = [\Psi^T(\Omega_1), ..., \Psi^T(\Omega_i), ..., \Psi^T(\Omega_N)]^T$, $\text{vec}(\dot{q}_{od}) = [\dot{q}_{od}^T, ..., \dot{q}_{od}^T, ..., \dot{q}_{od}^T]^T$, and $\text{vec}(\Delta_i) = [\Delta_1^T, ..., \Delta_i^T, ..., \Delta_N^T]^T$. Therefore, near an equilibrium point q_o, which can be either q_d or q_c, we have

$$\dot{q} = -k_0 u_{od}^2\ \partial\Psi_q(q, q_d)/\partial q\big|_{q=q_o}(q - q_o) + \text{vec}(\dot{q}_{od}) + \text{vec}(\Delta_i) \tag{7.74}$$

where the (i^{th}, j^{th}) element of the matrix $\frac{\partial \Psi_q(q,q_d)}{\partial q}$ is $\bigwedge_{ij} = \frac{\partial \Psi(\Omega_i)}{\partial \Omega_i}\frac{\partial \Omega_i}{\partial q_j}, (i,j) \in \mathbb{N}$ with $\mathbb{N}$ being the set of all agents. A simple calculation shows that

$$\frac{\partial \Omega_i}{\partial q_i} = \Big(1 + \sum_{i \in \mathbb{N}_i} \beta'_{ij}\Big) I_n + \sum_{j \in \mathbb{N}_i} \beta''_{ij} q_{ij} q_{ij}^T,\ \frac{\partial \Omega_i}{\partial q_j} = -\beta'_{ij} I_n - \beta''_{ij} q_{ij} q_{ij}^T. \tag{7.75}$$

for all $i = 1, ..., N, j \in \mathbb{N}_i, j \neq i$, where I_n denotes the identity matrix of size n. Let $\mathbb{N}^*$ be the set of the agents such that if the agents i and j belong to the set $\mathbb{N}^*$ then $\|q_{ij}\| < b_{ij}$. Next we will show that q_d is asymptotically stable and that q_c is unstable.

-Proof of q_d being asymptotic stable. Using properties of β_{ij} and ψ listed in (7.34) and (7.38), we have from (7.75) that for all $i = 1, ..., N, i \neq j$:

$$\begin{aligned}
&\frac{\partial \Psi(\Omega_i)}{\partial \Omega_i}\bigg|_{q=q_d} = I_n, \beta'_{ijd} = 0,\ \frac{\partial \Omega_i}{\partial q_i}\bigg|_{q=q_d} = I_n + \sum_{j \in \mathbb{N}_i} \beta''_{ijd} q_{ijd} q_{ijd}^T,\\
&\frac{\partial \Omega_i}{\partial q_j}\bigg|_{q=q_d} = -\beta''_{ijd} q_{ijd} q_{ijd}^T
\end{aligned} \tag{7.76}$$

where $\beta'_{ijd} = \beta'_{ij}\big|_{q_{ij}=q_{ijd}}$ and $\beta''_{ijd} = \beta''_{ij}\big|_{q_{ij}=q_{ijd}}$, with $q_{ijd} = q_{id} - q_{jd}$. We consider the function

$$V_d = \sqrt{1 + \|q - q_d\|^2} - 1 \tag{7.77}$$

whose derivative along the solutions of (7.74) with q_o replaced by q_d, using (7.76), and noting that $\dot{q}_{od} = \dot{q}_{id}$ satisfies

$$\begin{aligned}
\dot{V}_d = &\frac{1}{\sqrt{1 + \|q - q_d\|^2}}\Big(- k_0 u_{od}^2 \sum_{i=1}^{N} \|q_i - q_{id}\|^2 - \\
&k_0 u_{od}^2 \sum_{(i,j) \in \mathbb{N}, i \neq j} \beta''_{ijd}\big(q_{ijd}^T(q_{ij} - q_{ijd})\big)^2 + \sum_{i=1}^{N}(q_i - q_{id})^T \Delta_i\Big)\\
\leq &-\frac{k_0 u_{od}^2}{\sqrt{1 + \|q - q_d\|^2}} \sum_{i=1}^{N} \|q_i - q_{id}\|^2 + \sum_{i=1}^{N} \|\Delta_i\|
\end{aligned} \tag{7.78}$$

since $\beta''_{ijd} \geq 0$, see Property 1) in (7.34). The last inequality of (7.78) implies that q_d is asymptotically stable because $\lim_{t\to\infty} u_{od}^2(t) \neq 0$, and we have already proved that $\lim_{t\to\infty} \Delta_i(t) = 0$.

- *Proof of q_c being unstable.* Again using properties of β_{ij} and ψ in (7.34) and (7.38), we have from (7.75) that

$$\left.\frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\right|_{q=q_c} = I_n,\ \left.\frac{\partial\Omega_i}{\partial q_i}\right|_{q=q_c} = (1+\sum_{j\in\mathbb{N}_i}\beta'_{ijc})I_n + \sum_{j\in\mathbb{N}_i}\beta''_{ijc}q_{ijc}q_{ijc}^T,$$
$$\left.\frac{\partial\Omega_i}{\partial q_j}\right|_{q=q_c} = -\beta'_{ijc} - \beta''_{ijc}q_{ijc}q_{ijc}^T \tag{7.79}$$

for all $i = 1,...,N, i \neq j$, where $q_{ijc} = q_{ic} - q_{jc}$, $\beta'_{ijc} = \beta'_{ij}\big|_{q_{ij}=q_{ijc}}$ and $\beta''_{ijc} = \beta''_{ij}\big|_{q_{ij}=q_{ijc}}$. Since the related collision avoidance functions β_i, see (7.32), are specified in terms of relative distances between agents and it is extremely difficult to obtain q_c explicitly by solving (7.71), it is very difficult to use the Lyapunov function candidate $V_c = 0.5\|q - q_c\|$ to investigate stability of (7.74) at q_c. Therefore, we consider the Lyapunov function candidate

$$\bar{V}_c = \sqrt{1 + \|\bar{q} - \bar{q}_c\|^2} - 1 \tag{7.80}$$

where $\bar{q} = [q_{12}^T, q_{13}^T, ...q_{1N}^T, q_{23}^T, ..., q_{2N}^T, ..., q_{N-1,N}^T]^T$ and $\bar{q}_c = [q_{12c}^T, q_{13c}^T, ...q_{1Nc}^T, q_{23c}^T, ..., q_{2Nc}^T, ..., q_{N-1,Nc}^T]^T$. Differentiating both sides of (7.80) along the solution of (7.74) with q_o replaced by q_c gives

$$\begin{aligned}\dot{\bar{V}}_c = &-\frac{k_0u_{od}^2}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbb{N}\setminus\mathbb{N}^*}\|q_{ij}-q_{ijc}\|^2 - \frac{k_0u_{od}^2}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbb{N}^*}(1+\\ &N\beta'_{ijc})\|q_{ij}-q_{ijc}\|^2 - \frac{k_0u_{od}^2N}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbb{N}^*}\beta''_{ijc}\big(q_{ijc}^T(q_{ij}-q_{ijc})\big)^2 + \\ &\frac{1}{\sqrt{1+\|\bar{q}-\bar{q}_c\|^2}}\sum_{(i,j)\in\mathbb{N}}(q_{ij}-q_{ijc})^T(\Delta_i-\Delta_j)\end{aligned} \tag{7.81}$$

where $i \neq j$ and (7.79) has been used. To investigate stability properties of $\bar{q}_c$ based on (7.81), we will use (7.71). Define $\Omega_{ijc} = \Omega_{ic} - \Omega_{jc}$, $\forall(i,j) \in \{1,...,N\}$, $i \neq j$ where $\Omega_{ic} = \Omega_i|_{q=q_c} = 0$, see (7.71). Therefore $\Omega_{ijc} = 0$. Hence $\sum_{(i,j)\in\mathbb{N}^*} q_{ijc}^T\Omega_{ijc} = 0, i \neq j$, which by using (7.71) is expanded to

$$\begin{aligned}&\sum_{(i,j)\in\mathbb{N}^*}\big(q_{ijc}^T(q_{ijc}-q_{ijd}) + N\beta'_{ijc}q_{ijc}^Tq_{ijc}\big) = 0 \\ \Rightarrow &\sum_{(i,j)\in\mathbb{N}^*}(1+N\beta'_{ijc})q_{ijc}^Tq_{ijc} = \sum_{(i,j)\in\mathbb{N}^*}q_{ijc}^Tq_{ijd}\end{aligned} \tag{7.82}$$

where $i \neq j$. The sum $\sum_{(i,j)\in\mathbb{N}^*} q_{ijc}^T q_{ijd}$ is strictly negative since at the point, say F, where $q_{ij} = q_{ijd}$, $\forall (i,j) \in \mathbb{N}^*, i \neq j$ all attractive and repulsive forces are equal to zero while at the point, say C, where $q_{ij} = q_{ijc}$ $\forall (i,j) \in \mathbb{N}^*, i \neq j$ the sum of attractive and repulsive forces are equal to zero (but attractive and repulsive forces are nonzero). Therefore the point, say O, where $q_{ij} = 0$, $\forall (i,j) \in \mathbb{N}^*, i \neq j$ must locate between the points F and C for all $(i,j) \in \mathbb{N}^*, i \neq j$. That is there exists a strictly positive constant b such that $\sum_{(i,j)\in\mathbb{N}^*} q_{ijc}^T q_{ijd} < -b$, which is substituted into (7.82) to yield

$$\sum_{(i,j)\in\mathbb{N}^*} (1 + N\beta'_{ijc}) q_{ijc}^T q_{ijc} < -b, i \neq j. \tag{7.83}$$

Since $q_{ijc}^T q_{ijc} > 0, \forall (i,j) \in \mathbb{N}^*, i \neq j$, there exists a nonempty set $\mathbb{N}^{**} \subset \mathbb{N}^*$ such that for all $(i,j) \in \mathbb{N}^{**}, i \neq j$, $(1 + N\beta'_{ijc})$ is strictly negative, i.e. there exists a strictly positive constant b^{**} such that $(1 + N\beta'_{ijc}) < -b^{**}$, $\forall (i,j) \in \mathbb{N}^{**}, i \neq j$. We now write (7.81) as

$$\begin{aligned}
\dot{V}_c = &-\frac{k_0 u_{od}^2}{\sqrt{1 + \|\bar{q} - \bar{q}_c\|^2}} \Bigg[\sum_{(i,j)\in\mathbb{N}\setminus\mathbb{N}^*} \|q_{ij} - q_{ijc}\|^2 + \\
&\sum_{(i,j)\in\mathbb{N}^*\setminus\mathbb{N}^{**}} (1 + N\beta'_{ijc})\|q_{ij} - q_{ijc}\|^2 + N \sum_{(i,j)\in\mathbb{N}^*} \beta''_{ijc}\big(q_{ijc}^T(q_{ij} - q_{ijc})\big)^2 \Bigg] - \\
&\frac{k_0 u_{od}^2}{\sqrt{1 + \|\bar{q} - \bar{q}_c\|^2}} \sum_{(i,j)\in\mathbb{N}^{**}} (1 + N\beta'_{ijc})\|q_{ij} - q_{ijc}\|^2 + \\
&\frac{1}{\sqrt{1 + \|\bar{q} - \bar{q}_c\|^2}} \sum_{(i,j)\in\mathbb{N}} (q_{ij} - q_{ijc})^T (\Delta_i - \Delta_j)
\end{aligned} \tag{7.84}$$

where $i \neq j$. We now define a subspace such that $q_{ij} - q_{ijc} = 0$, $\forall (i,j) \in \mathbb{N} \setminus \mathbb{N}^{**}$ and $q_{ijc}^T(q_{ij} - q_{ijc}) = 0$, $\forall (i,j) \in \mathbb{N}^*, i \neq j$. In this subspace, we have

$$\begin{aligned}
\bar{V}_c &= \sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2} - 1, \\
\dot{\bar{V}}_c &= -\frac{k_0 u_{od}^2 \sum_{(i,j)\in\mathbb{N}^{**}} (1 + N\beta'_{ijc})\|q_{ij} - q_{ijc}\|^2}{\sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2}} + \\
&\quad \frac{\sum_{(i,j)\in\mathbb{N}^{**}} (q_{ij} - q_{ijc})^T (\Delta_i - \Delta_j)}{\sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2}}
\end{aligned} \tag{7.85}$$

Using $(1 + N\beta'_{ijc}) < -b^{**}$, $\forall (i,j) \in \mathbb{N}^{**}, i \neq j$, we can write (7.85) as

$$\bar{V}_c = \sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij} - q_{ijc}\|^2} - 1,$$
$$\dot{\bar{V}}_c \geq b^{**}k_0 u_{od}^2 \bar{V}_c - \sum_{(i,j)\in\mathbb{N}^{**}} \|\Delta_i - \Delta_j\|. \tag{7.86}$$

Now assume that q_c is a stable equilibrium point, i.e. $\lim_{t\to\infty}\|q_i(t) - q_{ic}\| = d_i, \forall i \in \mathbb{N}$ with d_i a nonnegative constant. Note that $\mathbb{N}^{**} \subset \mathbb{N}$, we have $\lim_{t\to\infty}\|q_i(t) - q_{ic}\| = d_i, \forall i \in \mathbb{N}^{**}$, which implies that $\lim_{t\to\infty}\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}, \forall (i,j) \in \mathbb{N}^{**}, i \neq j$ with d^{**} a nonnegative constant, since $q_{ij} = q_i - q_j$ and $q_{ijc} = q_{ic} - q_{jc}$. We now consider two cases: $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| \neq 0$ and $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| = 0$.

Case I: $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| \neq 0$. Since $\lim_{t\to\infty} u_{od}^2(t) \neq 0$ (Assumption 1) and we have already shown that $\lim_{t\to\infty}\Delta_i(t) = 0, \forall i \in \mathbb{N}$, $\bar{V}_c$ in (7.86) is divergent. Therefore, $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\|$ cannot tend to a constant but must be divergent. This contradicts $\lim_{t\to\infty}\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}$, i.e. q_c cannot be a set of stable equilibrium points but must be a set of an unstable ones in this case.

Case II: $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t_0) - q_{ijc}\| = 0$. There would be no contradiction. However this case is never observed in practice since the ever-present physical noise would cause $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\|$ to be different from 0 at the time $t^* > t_0$. We now need to show that once the sum $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\|$ is different from zero, this sum will not come back zero again for all $t \geq t^*$, i.e. the set of undesired equilibrium points q_c is not attractive. To do so, consider (7.86) with the initial time t^* instead of t_0, i.e.

$$\bar{V}_c(t) = \sqrt{1 + \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\|^2} - 1,$$
$$\dot{\bar{V}}_c(t) \geq b^{**}k_0 u_{od}^2 \bar{V}_c(t) - \sum_{(i,j)\in\mathbb{N}^{**}} \|\Delta_i(t) - \Delta_j(t)\|$$
$$\text{for } t \geq t^* \text{ and } \sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t^*) - q_{ijc}\| \geq \delta^* \tag{7.87}$$

where δ^* is a positive constant. Since $\lim_{t\to\infty} u_{od}^2(t) \neq 0$ (Assumption 1) and we have already shown that $\lim_{t\to\infty}\Delta_i(t) = 0, \forall i \in \mathbb{N}$, $\bar{V}_c$ in (7.87) is divergent for $t \geq t^*$. Therefore, $\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\|$, for $t \geq t^*$, cannot tend to a constant but must be divergent. This contradicts $\lim_{t\to\infty}\sum_{(i,j)\in\mathbb{N}^{**}} \|q_{ij}(t) - q_{ijc}\| = d^{**}$, i.e. q_c must also be a set of unstable ones point in this case. Proof of Theorem 7.7 is completed. □

7.5 Simulations

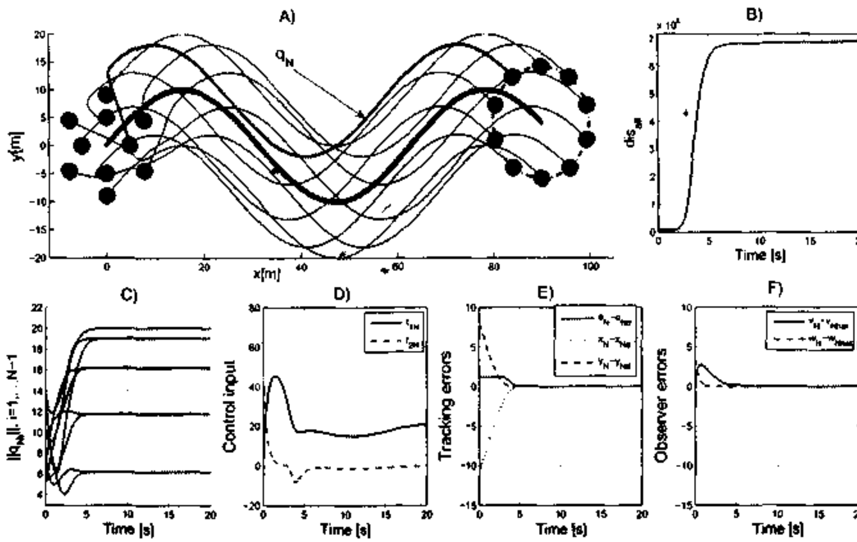


Fig. 7.3. Simulation results with 10 robots.

In this section, we simulate formation control of a group of $N = 10$ mobile robots to illustrate the effectiveness of the proposed controller. The physical parameters are taken from Section 1.3.1, Chapter 1, and $\underline{R}_i = 1.8, \overline{R}_i = 5, i = 1, ..., N$. The robots are initialized as follows: $x_i = R_0 \sin((i-1)2\pi/6), y_i = R_0 \cos((i-1)2\pi/6), \omega_{1i} = 0, \omega_{2i} = 0$, where $R_0 = 9$ for $i = 1, ..., 6$ and $R_0 = 5$ for $i = 7, ..., N$. The initial values of $\phi_i, i = 1, ..., N$ are taken as random numbers between 0 and 2π. The initial values of velocity estimates of the robots are taken as follows: $\hat{\omega}_{1i} = \hat{\omega}_{2i} = 0.1rad/s$. The reference trajectories are taken as $q_{od} = [s \;\; 10\sin(0.1s)]^T, \dot{s} = 1.5$ and $l_i = 10[\sin((i-1)2\pi/N) \;\; \cos((i-1)2\pi/N)]^T$. This choice of the reference trajectories means that the common reference trajectory q_{od} forms a sinusoidal trajectory, and that the desired formation configuration is a polygon whose vertices uniformly distribute on a circle centered on the common reference trajectory and with a radius 10. The functions $\beta_{ij}, (i,j) \in \mathbb{N}, i \neq j$ are taken as in (7.33) with $p = 4, a_{ij} = 2\underline{R}_i, b_{ij} = \overline{R}_i$. The function $\psi()$ is taken as arctan(). The control gains are chosen as $k_0 = 0.1, k_i = 2, L_i = 4I_2$ with I_2 an identify matrix of size 2. Indeed, the above choice of k_0 satisfies condition (7.39). Simulation results are plotted in Fig.7.3. It is seen that all robots nicely track their reference trajectories. During the first few seconds, the robots quickly move away from each other to avoid collisions then track their desired reference trajectory, see sub-figure A) in Fig.7.3, where the trajectory of the robot N is plotted in the thick line. In sub-figure B), we plot product of all gaps between robots: $Gap_{all} = \prod_{(i,j)\in\mathbb{N},i\neq j}(\|q_{ij}\|-(R_i+R_j))$. It is seen that Gap_{all} is always larger than zero. This means that $(\|q_{ij}\|-(R_i+R_j) > 0, \forall(i,j)\in\mathbb{N}, t\geq 0$, i.e. no collisions between any agents occurred. The distances between the robot N and other robots are plotted in sub-figure C) of Fig.7.3. Clearly, these distances are always larger than $\underline{R}_N + \underline{R}_i = 3.6, i = 1, ..., N-1$, i.e. there are no collisions between the robot N and all other robots in the group. The control inputs $[\tau_{1N}\ \tau_{2N}]^T$ of the robot N are plotted in sub-figure D) of Fig.7.3. Sub-figure E) in Fig.7.3 plots the tracking errors $x_N - x_{Nd}, y_N - y_{Nd}, \phi_N - \phi_{Nd}$ of the robot N. Indeed these errors tend to zero asymptotically. Sub-figure F) plots the observer errors $\tilde{v}_N$ and $\tilde{w}_N$. It is seen that these errors asymptotically converge to zero because the interlaced term Ξ_i for all $i \in \mathbb{N}$. For clarity, we only plot the results for the first 20 seconds in sub-figures B), C), D) E) and F).

7.6 Notes and references

Output-feedback tracking control of land, air, and sea vehicles has been solved for the case of fully actuated [17], pp. 311−334. The main difficulty of designing an observer-based output feedback for Lagrange systems in general is because of the Coriolis matrix, which results in quadratic cross terms of unmeasured velocities. In addition, nonholonomic constraints of mobile robots make the output-feedback problem challenging. For example, many solutions proposed for robot manipulator control ([17] and references therein) cannot directly be applied. Recently, output-feedback tracking of mobile robots was solved in [16]. In this work, based on a special coordinate transformation the exponential observer is designed to estimate the robot velocities. The control design is then based on the popular backstepping technique [12]. Some other results on output-feedback control of the single-DOF Lagrange systems were addressed in [19] (high-gain control), [11], and [20] for a nonlinear benchmark system. In general, nonlinear damping terms [12] are usually used to deal with observer errors in output feedback control control of nonlinear systems. The purpose of these nonlinear damping terms is to dominate nonlinearities multiplied by the observer errors. However, for formation control problem addressed in this chapter it is impossible to include nonlinear damping terms to do the job due to nonholonomic constraints and more importantly collision avoidance between the robots taken into account. Therefore, the interlaced observer is essential for formation control design in this chapter. The material in this chapter is based on the work in [62] and [63].

A

Mathematical Tools

This appendix presents necessary mathematical tools, which are used in the control design and stability analysis in the book. Some standard theorems, lemmas and corollaries, which are available in references, are sometimes given without a proof.

A.1 Lyapunov stability

Stability theory is important in system theory and engineering. There are various types of stability problems that arise in the study of dynamical systems. This section is concerned with stability of equilibrium points. Stability of equilibrium points is often characterized in the sense of Lyapunov, a Russian mathematician who laid the foundation of the theory, which now carries his name. Roughly speaking, an equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise it is unstable. It is asymptotically stable if in addition, all solutions tend to the equilibrium point as time approaches infinity. These kinds of notations will be mathematically made in this section. The material in this section is mainly taken from [58], [12] and [64].

Consider the following nonautonomous system

$$\dot{x} = f(t, x) \tag{A.1}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is a piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$ and $D \in \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

A.1.1 Definitions

Definition A.1. *The origin $x = 0$ is the equilibrium point of (A.1) if*

$$f(t,0) = 0, \ \forall\, t \geq 0. \tag{A.2}$$

Definition A.2. *A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition A.3. *A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to class KL if, for each fixed s, the mapping $\beta(r,s)$ belongs to class K with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.*

Definition A.4. *The equilibrium point $x = 0$ of (A.1) is*

1) stable if, for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \ \forall t \geq t_0 \geq 0 \tag{A.3}$$

2) uniformly stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ independent of t_0 such that (A.3) is satisfied

3)unstable if it is not stable

4)asymptotically stable if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$

5)uniformly asymptotically stable if it is uniformly stable and there is a positive constant c, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that

$$\|x(t)\| < \eta, \ \forall\, t \geq t_0 + T(\eta), \ \forall \, \|x(t_0)\| < c \tag{A.4}$$

6)globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$, and, for each pair of positive numbers η and c, there is $T = T(\eta, c) > 0$ such that

$$\|x(t)\| < \eta, \ \forall\, t \geq t_0 + T(\eta, c), \ \forall \, \|x(t_0)\| < c. \tag{A.5}$$

Definition A.5. *The equilibrium point $x = 0$ of (A.1) is exponentially stable if there exist positive constants c, k and λ such that*

$$\|x(t)\| \leq k \, \|x(t_0)\| \, e^{-\lambda(t-t_0)}, \ \forall\, t \geq t_0 \geq 0, \ \forall \ \|x(t_0)\| < c \tag{A.6}$$

and globally exponentially stable if (A.6) is satisfied for any initial state $x(t_0)$.

Definition A.6. *The equilibrium point $x = 0$ of (A.1) is K-exponentially stable if there exist positive constants c and λ and a class K function α such that*

$$\|x(t)\| \le \alpha\left(\|x(t_0)\|\right) e^{-\lambda(t-t_0)},\ \forall t \ge t_0 \ge 0,\ \forall\ \|x(t_0)\| < c \tag{A.7}$$

and globally K-exponentially stable if (A.7) is satisfied for any initial state $x(t_0)$.

Definition A.7. *The solutions of (A.1) are*

1) uniformly bounded if there exists a positive constant c, independent of $t_0 \ge 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of $t_0 \ge 0$, such that

$$\|x(t_0)\| \le a \ \Rightarrow \ \|x(t)\| \le \beta,\ \forall t \ge t_0 \tag{A.8}$$

2) globally uniformly bounded if (A.8) holds for arbitrarily large a.

3) uniformly ultimately bounded with ultimate bound b if there exist positive constants b and c, independent of $t_0 \ge 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \ge 0$, independent of $t_0 \ge 0$, such that

$$\|x(t_0)\| \le a \ \Rightarrow \ \|x(t)\| \le b,\ \forall t \ge t_0 + T \tag{A.9}$$

4) globally uniformly ultimately bounded if (A.9) holds for arbitrarily large a.

Definition A.8. *The system*

$$\dot{x} = f(t, x, u) \tag{A.10}$$

where f is piecewise continuous in t and locally Lipschitz in x and u, is said to be input-to-state stable (ISS) if there exists a class KL function β and a class K function γ, such that, for any input $u(\cdot)$ continuous and bounded on $[0, \infty)$, the solution exists for all $t \ge t_0 \ge 0$ and satisfies

$$\|x(t)\| \le \beta\left(\|x(t_0)\|, t - t_0\right) + \gamma\left(\sup_{t_0 \le \tau \le t} \|u(\tau)\|\right). \tag{A.11}$$

A.1.2 Lemmas and theorems

The following lemma provides equivalent definitions of uniform stability and uniform asymptotic stability by using class K and class KL functions.

Lemma A.9. *The equilibrium point $x = 0$ of (A.1) is*

1) uniformly stable if and only if there exist a class K function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \alpha\left(\|x(t_0)\|\right), \ \forall t \geq t_0 \geq 0, \ \forall \ \|x(t_0)\| < c \tag{A.12}$$

2) uniformly asymptotically stable if and only if there exist a class KL function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right), \ \forall t \geq t_0 \geq 0, \ \forall \ \|x(t_0)\| < c \tag{A.13}$$

3) globally uniformly asymptotically stable if and only if inequality (A.13) is satisfied for any initial state $x(t_0)$.

Proof of Lemma A.9. See [58].

Lemma A.10. *Assume that $d : \mathbb{R}^n \to \mathbb{R}^n$ satisfies*

$$P\left[\frac{\partial d}{\partial x}\right] + \left[\frac{\partial d}{\partial x}\right]^T P \geq 0, \ \forall \, x \in \Re^n \tag{A.14}$$

where $P = P^T > 0$. Then

$$(x - y)^T P(d(x) - d(y)) \geq 0, \quad \forall x, y \in \Re^n. \tag{A.15}$$

Proof of Lemma A.10. See [65].

Lemma A.11. *The following nonlinear interconnected system*

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2) + g_1(t, x_1, x_2)u, \\ \dot{x}_2 &= f_2(t, x_1, x_2) + g_2(t, x_1, x_2)u \end{aligned} \tag{A.16}$$

where $x_i \in \mathbb{R}$, $i = 1, 2$, $f_i(t, x_1, x_2)$ are locally Lipschitz in x_i and piecewise continuous in t; $u \in \mathbb{R}$ is the control input, and $g_2(t, x_1, x_2) \neq 0$, $\forall \, t \geq 0$, $x_i \in \mathbb{R}$, can be transformed to the following system

$$\begin{aligned} \dot{z}_1 &= \gamma_1(t, z_1, x_2), \\ \dot{x}_2 &= \gamma_2(t, z_1, x_2) + \varphi_2(t, z_1, x_2)u \end{aligned} \tag{A.17}$$

Proof of Lemma A.11. Define

$$z_1 = x_1 + \pi(t, x_1, x_2) \tag{A.18}$$

where $\pi(t, x_1, x_2)$ is to be determined. Differentiating both sides of (A.18) along the solutions of (A.17) yields

$$\dot{z}_1 = \left(\frac{\partial\pi(t,x_1,x_2)}{\partial x_1} + 1\right) f_1(t, x_1, x_2) + \frac{\partial\pi(t,x_1,x_2)}{\partial x_2} f_2(t, x_1, x_2) + \frac{\partial\pi(t,x_1,x_2)}{\partial t} + \left(\left(\frac{\partial\pi(t,x_1,x_2)}{\partial x_1} + 1\right) g_1(t, x_1, x_2) + \frac{\partial\pi(t,x_1,x_2)}{\partial x_2} g_2(t, x_1, x_2)\right) u. \tag{A.19}$$

Now choosing the function $\pi(t, x_1, x_2)$ such that

$$\left(\left(\frac{\partial\pi(t,x_1,x_2)}{\partial x_1} + 1\right) g_1(t, x_1, x_2) + \frac{\partial\pi(t,x_1,x_2)}{\partial x_2} g_2(t, x_1, x_2)\right) = 0 \tag{A.20}$$

results in (A.18), where the functions $\gamma_i(t, z_1, x_2)$ and $\varphi_2(t, z_1, x_2)$ are defined as

$$\begin{aligned}
\gamma_1(t, z_1, x_2) &:= \left(\frac{\partial\pi(t,x_1,x_2)}{\partial x_1} + 1\right) f_1(t, x_1, x_2) + \frac{\partial\pi(t,x_1,x_2)}{\partial x_2} f_2(t, x_1, x_2) + \frac{\partial\pi(t,x_1,x_2)}{\partial t}, \\
\gamma_2(t, z_1, x_2) &:= f_2(t, x_1, x_2), \\
\varphi_2(t, z_1, x_2) &:= g_2(t, x_1, x_2)
\end{aligned} \tag{A.21}$$

with x_1 being solved from (A.18) and substituted in.

Remark A.12. 1). The success of the above lemma depends on the possibility of finding a solution to the partial differential equation (A.21). Solving this partial differential equation is difficult in general but might be simple in some specific cases such as strict-forward second order systems and the ship systems in this thesis.

2). In some cases, designing a control input u for the transformed system (A.18) is simpler than for the original system (A.17).

The main Lyapunov stability theorem, which has a number of applications in studying stability of (A.1), is given below.

Theorem A.13. *Let $x = 0$ be an equilibrium point of (A.1) and $D = \left\{x \in \mathbb{R}^n \left|_{\|x\|<r}\right.\right\}$. Let $V : D \times \mathbb{R}^n \to \mathbb{R}^+$ be a continuously differentiable function such that $\forall t \geq 0$, $\forall x \in D$,*

$$\begin{aligned}
&\gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|), \\
&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\gamma_3(\|x\|).
\end{aligned} \tag{A.22}$$

Then the equilibrium point is

1) uniformly stable, if γ_1 and γ_2 are class K functions on $[0,r)$ and $\gamma_3 \geq 0$ on $[0,r)$;

2) uniformly asymptotically stable, if γ_1, γ_2 and γ_3 are class K functions on $[0,r)$;

3) exponentially stable if $\gamma_i(\rho) = k_i\rho^\alpha$ on $[0,r)$, $k_i > 0$, $\alpha > 0$, $i = 1,2,3$;

4) globally uniformly stable if $D = \mathbb{R}^n$, γ_1 *and* γ_2 *are class* K_∞ *functions, and* $\gamma_3 \geq 0$ *on* $\mathbb{R}^+$;

5) globally uniformly asymptotically stable if $D = \mathbb{R}^n$, γ_1 *and* γ_2 *are class* K_∞ *functions, and* γ_3 *is a class* K *function on* $\Re^+$; *and*

6) globally exponentially stable, if $D = \mathbb{R}^n$, $\gamma_i(\rho) = k_i \rho^\alpha$ *on* $\mathbb{R}^+$, $k_i > 0, \alpha > 0, \ i = 1, 2, 3.$

Proof of Theorem A.13. See [12].

Theorem A.14. *Let* $x = 0$ *be an equilibrium point of (A.1). Let* $V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuously differentiable function such that*

$$\begin{aligned} &\gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|), \\ &\dot{V} = \tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial x} f(t,x) \leq -W(x) \leq 0. \end{aligned} \tag{A.23}$$

$\forall t \geq 0, \forall x \in \Re^n$, *where* γ_1 *and* γ_2 *are class* K_∞ *functions, and* W *is a continuous function. Then all solutions of (A.1) are globally uniformly bounded and satisfy*

$$\lim_{t \to \infty} W(x(t)) = 0. \tag{A.24}$$

In addition, if $W(x)$ *is positive definite, then the equilibrium point* $x = 0$ *is globally uniformly asymptotically stable.*

Proof of Theorem A.14. See [12].

The following Lyapunov-like theorem is useful for showing uniform boundedness and ultimate boundedness.

Theorem A.15. *Let* $D \subset \mathbb{R}^n$ *be a domain that contains the origin and* $V : [0, \infty) \times D \to \mathbb{R}$ *be a continuously differentiable function such that*

$$\begin{aligned} &\alpha_1(\|x\|) \leq V(x,t) \leq \alpha_2(\|x\|), \\ &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x), \ \forall \|x\| \geq \mu > 0 \end{aligned} \tag{A.25}$$

$\forall t \geq 0, \forall x \in D$ *where* α_1 *and* α_2 *are class* K *functions, and* W *is a continuous positive definite function. Take* $r > 0$ *such that* $B_r \subset D$ *and suppose that*

$$\mu < \alpha_2^{-1}(\alpha_1(r)). \tag{A.26}$$

Then there exists a class KL *function* β *and for every initial state* $x(t_0)$, *satisfying* $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$, *there is* $T \geq 0$ *(dependent on* $x(t_0)$ *and* μ*) such that the solution of (A.1) satisfies*

$$\begin{aligned} &\|x(t)\| \leq \beta\left(\|x(t_0)\|, t-t_0\right), \ \forall\ t_0 \leq t \leq t_0+T, \\ &\|x(t)\| < \alpha_2^{-1}(\alpha_1(r)), \ \forall\ t \geq t_0+T. \end{aligned} \tag{A.27}$$

Moreover, if $D = \mathbb{R}^n$ and α_1 belongs to class K_∞, then (A.27) holds for any initial state $x(t_0)$ with no restriction on how large μ is.

Proof of Theorem A.15. See [58].

A.1.3 Stability of cascade systems

Consider the following cascade system

$$\begin{aligned} \dot{z}_1 &= f_1(t, z_1) + g(t, z_1, z_2) z_2, \\ \dot{z}_2 &= f_2(t, z_2) \end{aligned} \tag{A.28}$$

where $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^m$, $f_1(t, z_1)$ is continuously differentiable in (t, z_1) and $f_2(t, z_2)$, $g(t, z_1, z_2)$ are continuous, and locally Lipschitz in z_2 and (z_1, z_2) respectively.

If we set $z_2 = 0$ in the first equation of (A.28) becomes $\dot{z}_1 = f_1(t, z_1)$. Therefore we can view the first equation of (A.28) as the system

$$\Omega_1 : \dot{z}_1 = f_1(t, z_1) \tag{A.29}$$

which is perturbed by the output of the system

$$\Omega_2 : \dot{z}_2 = f_2(t, z_2). \tag{A.30}$$

Now assume that the systems Ω_1 and Ω_2 are asymptotically stable at the origin, i.e. (A.29) and (A.30) yield $\lim_{t\to\infty} z_1(t) = 0$ and $\lim_{t\to\infty} z_2(t) = 0$, respectively. Based on these assumptions, it is plausible to conclude that the system (A.28) is asymptotically stable at the origin in general. In many cases, the solution $z_1(t)$ of the system (A.28) goes to infinity in finite time. This phenomenon can be seen from the following simple cascade system:

$$\begin{aligned} \dot{z}_1 &= -k_1 z_1 + z_1^2 z_2, \\ \dot{z}_2 &= -k_2 z_2 \end{aligned} \tag{A.31}$$

where k_1 and k_2 are strictly positive constants. It is obvious that the subsystems $\dot{z}_1 = -k_1 z_1$ and $\dot{z}_2 = -k_2 z_2$ are globally exponentially stable at the origin. It is straightforward to show that the solution of (A.31) is

$$\begin{aligned} z_1(t) &= \frac{z_1(t_0)(k_1+k_2)}{z_1(t_0)z_2(t_0)e^{-k_2(t-t_0)} + (k_1+k_2 - z_1(t_0)z_2(t_0))e^{k_1(t-t_0)}}, \\ z_2(t) &= z_2(t_0)e^{-k_2(t-t_0)}. \end{aligned} \tag{A.32}$$

It is seen from (A.32) that if $z_1(t_0)z_2(t_0) < k_1 + k_2$, then both $z_1(t)$ and $z_2(t)$ are bounded and converge asymptotically and exponentially to zero, respectively. If $z_1(t_0)z_2(t_0) = k_1 + k_2$ then $z_2(t)$ is still bounded and converges exponentially to zero but $z_1(t)$ tends to infinity exponentially fast. If $z_1(t_0)z_2(t_0) > k_1 + k_2$, the situation is catastrophic, i.e. $z_1(t)$ tends to infinity when $t \to t_0 + t_f$ with

$$t_f = \frac{1}{k_1 + k_2} \ln \left(\frac{z_1(t_0)z_2(t_0)}{z_1(t_0)z_2(t_0) - (k_1 + k_2)} \right). \tag{A.33}$$

The following theorem gives sufficient conditions of stability of the cascade system (A.28) based on stability of (A.29) and (A.30) and the connected term $g(t, z_1, z_2)$.

Theorem A.16. *Consider the following assumptions:*

1) The systems (A.29) and (A.30) are both globally uniformly asymptotically stable (GUAS) and we know explicitly a C^1 Lyapunov function $V(t, z_1)$, two class K_∞ functions α_1 and α_2, a class K function α_4, and a positive semi-definite function $W(z_1)$ such that

$$\begin{array}{c} \alpha_1(\|z_1\|) \leq V(t, z_1) \leq \alpha_2(\|z_1\|), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1} f_1(t, z_1) \leq -W(z_1), \\ \left\| \frac{\partial V}{\partial z_1} \right\| \leq \alpha_4(\|z_1\|). \end{array} \tag{A.34}$$

2) For each fixed z_2 there exists a continuous function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{s\to\infty} \lambda(s) = 0$ such that

$$\left\| \frac{\partial V}{\partial z_1} g(t, z_1, z_2) \right\| \leq \lambda(\|z_1\|)\, W(z_1). \tag{A.35}$$

3) There exist continuous functions $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha_5 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(t, z_1, z_2)\| \leq \theta(\|z_2\|)\, \alpha_5(\|z_1\|) \tag{A.36}$$

and a continuous nondecreasing function $\alpha_6 : \mathbb{R}^+ \to \mathbb{R}^+$, and a nonnegative constant a such that

$$\begin{array}{l} \alpha_6(s) \geq \alpha_4\left(\alpha_1^{-1}(s)\right) \alpha_5\left(\alpha_1^{-1}(s)\right), \\ \int_a^\infty \frac{ds}{\alpha_6(s)} = \infty. \end{array} \tag{A.37}$$

4) For each $r > 0$, there exist constants $\chi > 0$ and $\eta > 0$ such that for all $t \geq 0$ and all $\|z_2\| < r$

$$\left\| \frac{\partial V}{\partial z_1} g(t, z_1, z_2) \right\| \leq \chi W(z_1), \forall \|z_1\| \geq \eta. \tag{A.38}$$

5) There exists a class K function ϕ such that the solution $z_2(t)$ of (A.30) satisfies

$$\int_{t_0}^{\infty} \|z_2(t)\| \leq \phi\left(\|z_2(t_0)\|\right). \tag{A.39}$$

Then we can conclude that if
Assumptions 1) and 2), or
Assumptions 1), 3) and 4), or
Assumptions 1), 3) and 5)
hold then the cascade system (A.28) is globally uniformly asymptotically stable.

Proof of Theorem A.16. See [66].

A.2 Input-to-state stability

The material in this section is mainly taken from [67].

Definition A.17. *The system*

$$\dot{x} = f(t, x, u) \tag{A.40}$$

where f is piecewise continuous in t and locally Lipchitz in x and u, is said to be input-to-state stable (ISS) if there exist a class KL function β and a class K function γ, such that for any $x(0)$ and for any input $u(.)$ continuous and bounded on $[0, \infty)$the solution exists for all $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma\left(\sup |u(\tau)|_{t_0 \leq \tau \leq t}\right) \tag{A.41}$$

for all t_0 and t such that $0 \leq t_0 \leq t$

The following theorem establishes the equivalence between the existence of a Lyapunov like function and the input-to-state stability.

Theorem A.18. *Suppose that for the system (A.40) there exists a C^1 function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$,*

$$\begin{aligned} &\gamma_1(|x|) \leq V(t, x) \leq \gamma_2(|x|) \\ &|x| \geq \rho(|u|) \;\Rightarrow \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\gamma_3(|x|) \end{aligned} \tag{A.42}$$

where γ_1, γ_2 and ρ are class K_∞ functions and γ_3 is a K class function. Then the system (A.40) is ISS with $\gamma = \gamma_1^{-1} \circ \gamma_2 \circ \rho$.

Proof. If $x(t_0)$ is in the set

$$Rt_0 = \{x \in IR^n \,||x| \le \rho\,(\sup|u(\tau)|_{\tau \ge t_0})\} \tag{A.43}$$

then $x(t)$ remains within the set

$$St_0 = \left\{x \in IR^n \,||x| \le \gamma_1^{-1} \circ \gamma_2 \circ \rho\,(\sup|u(\tau)|_{\tau \ge t_0})\right\} \tag{A.44}$$

for all $t \ge t_0$. Define $B = [t_0\,, T)$ as the time interval before $x(t)$ enters Rt_0 for the first time. In view of the definition of Rt_0 we have

$$\dot{V} \le -\gamma_3 \circ \gamma_2^{-1}(V) \forall t \in B \tag{A.45}$$

Then there exists a class KL function βv such that

$$V(t) \le \beta v(V(t_0), t - t_0) \forall t \in B \tag{A.46}$$

which implies

$$|x(t)| \le \gamma_1^{-1}(\beta v(\gamma_2(x(|t_0|), t - t_0)) \;=\; \beta(x(|t_0|), t - t_0), \quad \forall t \in B. \tag{A.47}$$

On the other hand by (A.44), we conclude that

$$|x(t)| \le \gamma_1^{-1} \circ \gamma_2 \circ \rho(\sup|u(\tau)|) \;=\; \gamma(\sup|u(\tau)|_{\tau \ge t_0}), \quad \forall t \in [t_0, \infty]\backslash B \tag{A.48}$$

Then by (A.47) and (A.48),

$$|x(t)| \le \beta(|x(t_0), t - t_0) + \gamma(\sup|u(\tau)|_{\tau \ge t_0}), \quad \forall t \ge t_0 \ge 0 \tag{A.49}$$

By causality, we have

$$|x(t)| \le \beta(|x(t_0), t - t_0) + \gamma(\sup|u(\tau)|_{t_0 \le \tau \le t}), \quad \forall t \ge t_0 \ge 0 \tag{A.50}$$

A function V satisfying conditions (A.42) is called an ISS Lyapunov function. □

A.3 Control Lyapunov functions (clf)

This section, taken from [12], presents an extension of the Lyapunov function concept, which is a useful tool to design an adaptive controller for nonlinear systems. Assuming that the problem is to design a feedback control law $\alpha(x)$ for the time-invariant system:

$$\dot{x} = f(x, u), \;\; x \in \mathbb{R}^n, \;\; u \in \mathbb{R}, \;\; f(0,0) = 0 \tag{A.51}$$

such that the equilibrium $x = 0$ of the closed loop system:

$$\dot{x} = f(x, \alpha(x)) \tag{A.52}$$

is globally asymptotically stable. We can take a function $V(x)$ as a Lyapunov candidate function, and require that its derivative along the solutions of (A.51) satisfy $\dot{V}(x) \leq -W(x)$, where $W(x)$ is a positive definite function. We therefore need to find $\alpha(x)$ to guarantee that for all $x \in \mathbb{R}^n$ such that

$$\frac{\partial V(x)}{\partial x} f(x, \alpha(x)) \leq -W(x) \tag{A.53}$$

This is a difficult problem. A stabilizing control law for (A.52) may exist but we may fail to satisfy (A.53) because of a poor choice of $V(x)$ and $W(x)$. A system for which a good choice of $V(x)$ and $W(x)$ exists is said to possess a Control Lyapunov Function (clf). For systems affine in the control:

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0 \tag{A.54}$$

the clf inequality (A.53) becomes

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)\alpha(x) \leq -W(x) \tag{A.55}$$

If $V(x)$ is a clf for (A.54), then a particular stabilizing control law $\alpha(x)$, smooth for all $x \neq 0$, is given

$$u = \alpha(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f(x) + \sqrt{\left(\frac{\partial V}{\partial x} f(x)\right)^2 + \left(\frac{\partial V}{\partial x} g(x)\right)^4}}{\frac{\partial V}{\partial x} g(x)}, & \frac{\partial V}{\partial x} g(x) \neq 0 \\ 0 & \frac{\partial V}{\partial x} g(x) = 0 \end{cases} \tag{A.56}$$

It should be noted that (A.55) can be satisfied only if

$$\frac{\partial V}{\partial x} g(x) = 0 \Rightarrow \frac{\partial V}{\partial x} f(x) < 0, \qquad \forall x \neq 0 \tag{A.57}$$

and that in this case (A.55) becomes

$$W(x) = \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} > 0, \quad \forall x \neq 0 \tag{A.58}$$

The main drawback of the clf concept as a design tool is that for most nonlinear systems a clf is not known. The task of finding a appropriate clf may be as complex as that of designing a stabilizing feedback law.

A.4 Backstepping

This section is taken from [12].

Assumption A.19 *Consider the system*

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0 \tag{A.59}$$

where $x \in \mathbb{R}^n$ is the state and u is the control input. There exist a continuously differentiable feedback control law

$$u = \alpha(x), \alpha(0) = 0 \tag{A.60}$$

and a smooth, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}\left[f(x) + g(x)\alpha(x)\right] \leq -W(x) \leq 0, \forall x \in IR^n \tag{A.61}$$

where $W : \mathbb{R}^n \to \mathbb{R}$ is positive semidefinite.

It should be noted that under this assumption, the control law (A.60), applied to the system (A.59), guarantees global boundedness of $x(t)$, and the regulation of $W(x) : \lim_{t\to\infty} W(x) = 0$. If $W(x)$ is positive definite, the control law (A.60) renders the global asymptotic stable equilibrium of (A.59).

Theorem A.20. *Let the system (A.59) be augmented by an integrator:*

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= u \end{aligned} \tag{A.62}$$

and suppose that the first equation of (A.62) satisfies Assumption A.19 with ξ as its control. If $W(x)$ is positive definite then

$$V_a = V(x) + \frac{1}{2}\left[\xi - \alpha(x)\right]^2 \tag{A.63}$$

is a clf for the system (A.62), that is, there exists a feedback control law $u = \alpha_a(x, \xi)$ which renders $x = 0$, $\xi = 0$ the GAS equilibrium of (A.62). One such control law choice is

$$u = -c(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}\left[f(x) + g(x)\xi\right] - \frac{\partial V}{\partial x} g(x), \quad c > 0 \tag{A.64}$$

If $W(x)$ is only positive semidefinite, then there exist a feedback control law which renders $\dot{V}_a \leq -W_a(x, \xi) \leq 0$, such that $W_a(x, \xi) > 0$ whenever $W(x) > 0$ or $\xi \neq \alpha(x)$. This guarantees global boundedness and convergence of $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ to the largest invariant set M_a contained in the set $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+1} \middle| W(x) = 0, \ \xi = \alpha(x) \right\}$.

Proof. We only prove the first part of the theorem. Proof of the second part is trivial. Introducing an error variable

$$z = \xi - \alpha(x) \tag{A.65}$$

and differentiating analytically (A.63), that is without the need of a differentiator, (A.62) can be written as

$$\begin{aligned} \dot{x} &= f(x) + g(x)[\alpha(x) + z] \\ \dot{z} &= u + \tfrac{\partial \alpha}{\partial x}\left[f(x) + g(x)(\alpha(x) + z)\right] \end{aligned} \tag{A.66}$$

Using (A.61), the first time derivative of (A.63) along the solution of (A.66) can be expressed as

$$\dot{V}_2 \leq -W(x) + z\left[u - \frac{\partial \alpha}{\partial x}\left(f(x) + g(x)(\alpha(x) + z)\right) + \frac{\partial V}{\partial x} g(x)\right] \tag{A.67}$$

Any control law, one such as (A.64), which renders $\dot{V}_a \leq -W_a(x,\xi) \leq -W(x)$, with W_a positive definite in z, guarantees global boundedness of x, z, and regulation of $W(x)$ and $z(t)$. □

Corollary A.21. *Let the system (A.62) satisfying Assumption A.19 with $\alpha(x) = \alpha_0(x)$ be augmented by a change of k integrators so that u is replaced by ξ_1, the state of the last integrator in the chain:*

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi} &= \xi_2 \\ &\vdots \\ \dot{\xi}_{k-1} &= \xi_k \\ \dot{\xi}_k &= u \end{aligned} \tag{A.68}$$

For this system, repeated application of Theorem A.20 with $\xi_1, \cdots, \xi_k$ as virtual controls, results in the Lyapunov function

$$\dot{V}(x, \xi_1, \cdots, \xi_k) = V(x) + \frac{1}{2}\sum_{i=1}^{k}\left[\xi_i - \alpha_{i-1}(x, \xi_1, \cdots, \xi_k)\right]^2 \tag{A.69}$$

Any choice of feedback control which renders $\dot{V}_a \leq -W_a(x, \xi_1, \cdots, \xi_k) \leq 0$, with $W_a(x, \xi_1, \cdots, \xi_k) = 0$ only if $W(x) = 0$ and $\xi_i \neq \alpha_{i-1}(x, \xi_1, \cdots, \xi_{i-1})$, i=1,...,k, guarantees that $\left[x^T(t), \xi_1, \cdots, \xi_k\right]^T$ is globally bounded and converges to the largest invariant set M_a contained in the set

$$E_a = \left\{\left[x^T, \xi_1, \cdots, \xi_k\right]^T \in \mathbb{R}^n \,|W(x) = 0, \, \xi_i = \alpha_{i-1}(x, \xi_1, \cdots, \xi_{i-1})\right\}$$

for all $i = 1, ..., k$. Furthermore, if W(x) is positive definite, that is, if x=0 can be rendered GAS through ξ_1, then (A.69) is a clf for (A.68) and the equilibrium $x = 0, \xi_1 = 0, ..., \xi_k = 0$ can be rendered GAS through u.

A.5 Stabilization in the presence of uncertainty

The power of adaptive control is exhibited in the presence of uncertain nonlinearities and unknown parameters. These uncertainties in linear systems make the control design procedure difficult and become more serious in control of nonlinear systems. For nonlinear systems, the states can easily escape to infinity in a finite time. This is referred to as finite escape. The following theorem introduces the use of a term in the control law called nonlinear damping to stabilize the system (A.59) in the presence of disturbance. The material in this section is based on [12].

Theorem A.22. *Consider the system (A.59) satisfying assumption 2.12 which is perturbed as*

$$\dot{x} = f(x) + g(x)\left[u + \varphi(x)^T \Delta(x,u,t)\right], \qquad f(0) = 0 \tag{A.70}$$

where $\varphi(x)$ *is a vector of known smooth nonlinear functions, and* $\Delta(x,u,t)$*is a vector of uncertain nonlinearities which are uniformly bounded for all values of* x, u, *and* t. *If W(x) is positive definite and radially unbounded, then the control*

$$u = \alpha(x) - k\frac{\partial V}{\partial x}(x)g(x)\left|\varphi(x)\right|^2, \quad k > 0 \tag{A.71}$$

when applied to the system (A.70) renders the closed-loop system to be stable with respect to the disturbance input $\Delta(x,u,t)$ *and hence guarantees global uniform boundedness of x(t) and convergence to the residual set*

$$R = \left\{ x : |x| \le \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k}\right)\right\}, \tag{A.72}$$

where γ_1, γ_2, γ_3 *are class* K_∞ *functions such that*

$$\begin{aligned} \gamma_1(|x|) &\le V(x) \le \gamma_2(|x|) \\ \gamma_3(|x|) &\le W(x) \end{aligned} \tag{A.73}$$

Proof. By using (A.71) and (A.73), the first time derivative of $V(x)$ is

$$\begin{aligned} \dot{V} &= \tfrac{\partial V}{\partial x}[f + gu] + \tfrac{\partial V}{\partial x} g\varphi^T \Delta \\ &= \tfrac{\partial V}{\partial x}[f + gu] - k\left(\tfrac{\partial V}{\partial x} g\right)^2 |\varphi|^2 + \tfrac{\partial V}{\partial x} g\varphi^T \Delta \\ &\le -W(x) - k\left(\tfrac{\partial V}{\partial x} g\right)^2 |\varphi|^2 + \tfrac{\partial V}{\partial x} g\varphi^T \Delta \\ &\le -W(x) - k\left(\tfrac{\partial V}{\partial x} g\right)^2 |\varphi|^2 + \tfrac{\partial V}{\partial x} g\,|\varphi|\,\|\Delta\|_\infty \\ &\le -W(x) + \tfrac{\|\Delta\|_\infty^2}{4k} \end{aligned} \tag{A.74}$$

From (A.74), it follows that $\dot{V}$ is negative whenever $W(x) \ge \frac{\|\Delta\|_\infty^2}{4k}$. Combining this with the second equation of (A.73), we conclude that

$$|x(t)| > \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k}\right) \Rightarrow \dot{V} < 0 \tag{A.75}$$

This means that if $|x(0)| \leq \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k}\right) \Rightarrow \dot{V} < 0$ then

$$V(x(t)) \leq \gamma_2 \circ \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k}\right) \tag{A.76}$$

which implies that

$$|x(t)| \leq \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k}\right) \tag{A.77}$$

If on the other hand $|x(0)| \geq \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k}\right)$ then $V(x) \leq V(x(0))$, which implies

$$|x(t)| \leq \gamma_1^{-1}\gamma_2\left(|x(0)|\right) \tag{A.78}$$

Combining (A.77) and (A.78) leads to the global uniform boundedness of x(t)

$$\|x\|_\infty \leq \max\left\{\gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k}\right), \gamma_1^{-1} \circ \gamma_2(|x(0)|\right\} \tag{A.79}$$

while (A.75) and the first equation of (A.71) prove the convergence of $x(t)$ to the residual set defined in (A.72). The ISS property of the closed loop system with respect to the disturbance $\Delta(x, u, t)$ input follows from Theorem A.18.□

Combining the above Theorem with Theorem A.20 leads to the following Corollary.

Corollary A.23. *Consider the following system*

$$\dot{x} = f(x) + g(x)u + F(x)^T \Delta_1(x, u, t) \tag{A.80}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $F(x)$ is an $(n \times q)$ matrix of known smooth nonlinear functions, and $\Delta_1(x, u, t)$ is a $(q\times 1)$ vector of uncertain nonlinearities which is uniformly bounded for all values of x, u, t. Suppose that there exists a feedback control law $u = \alpha(x)$ that renders $x(t)$ globally uniformly bounded bounded, and that this is established via positive definite and radially unbounded functions $V(x)$, $W(x)$, and a constant b such that

$$\frac{\partial V}{\partial x}\left[f(x) + g(x)\alpha(x) + F(x)^T \Delta_1(x, u, t)\right] \leq -W(x) + b \tag{A.81}$$

Now consider the augmented system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi + F(x)^T \Delta_1(x,u,t) \\ \dot{\xi} &= u + \varphi^T(x,\xi)\Delta_2(x,\xi,u,t) \end{aligned} \tag{A.82}$$

where $\varphi(x,\xi)$ is a $(q \times 1)$ vector of known smooth nonlinear functions, and $\Delta_2(x,u,\xi,t)$ is a $(q \times 1)$ vector of uncertain nonlinearities which is uniformly bounded for all values of x, u, ξ,t. For this system, the feedback control law

$$\begin{aligned} u = & -c(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}[f(x) + g(x)\alpha(x)] - \\ & \frac{\partial V}{\partial x} g(x) - k(\xi - \alpha(x)) \left\{ |\varphi(x,\xi)|^2 + \left| \frac{\partial \alpha}{\partial x} F(x) \right|^2 \right\} \end{aligned} \tag{A.83}$$

guarantees global uniform boundedness of $x(t)$ and $\xi(t)$ with any $c > 0$ and $k > 0$.

A.6 Barbalat like lemmas

This section presents lemmas that are useful in investigating convergence of time-varying systems.

Lemma A.24. *Consider the function $\phi : \mathbb{R}^+ \to \mathbb{R}$. If ϕ is uniformly continuous and $\lim_{t\to\infty} \int_0^\infty \phi(\tau)d\tau$ exists and finite, then*

$$\lim_{t\to\infty} \phi(t) = 0. \tag{A.84}$$

Proof of Lemma A.24. See [58].

Lemma A.25. *Assume that a nonnegative scalar differentiable function $f(t)$ enjoys the following conditions*

$$1) \quad \left| \frac{d}{dt} f(t) \right| \leq k_1 f(t), \forall t \geq 0, \quad and \quad 2) \quad \int_0^\infty f(t)dt \leq k_2 \tag{A.85}$$

where k_1 and k_2 are positive constants, then $\lim_{t\to\infty} f(t) = 0$.

It is noted that Lemma A.25 is different from Barbalat's lemma A.24. While Barbalat's lemma assumes that $f(t)$ is uniformly continuous, Lemma 2 assumes that $|\frac{d}{dt}f(t)|$ is bounded by $k_1 f(t)$.

Proof of Lemma A.25. Integrating both sides of 1) in (A.85) gives

$$\begin{aligned} f(t) &\leq f(0) + k_1 \int_0^t f(s)ds \leq f(0) + k_1 k_2, \\ f(t) &\geq f(0) - k_1 \int_0^t f(s)ds \geq f(0) - k_1 k_2. \end{aligned} \tag{A.86}$$

These inequalities imply that $f(t)$ is a uniform bounded function. From (A.86) and the second condition in (A.85), we have $f(t)$ is also bounded on the half axis $[0,\infty]$, i.e. $f(t) \le k_3$ with k_3 a positive constant. Hence $\left|\frac{d}{dt}f(t)\right| \le k_1k_3$. Now assume that $\lim_{t\to\infty} f(t) \ne 0$. Then there exists a sequence of points t_i and a positive constant ϵ such that $f(t_i) \ge \epsilon$, $t_i \to \infty$, $i \to \infty, |t_i - t_{i-1}| > 2\epsilon/(k_1k_3)$ and moreover $f(s) \ge \epsilon/2$, $s \in L_i = [t_i - \epsilon/(2k_1k_3), t_i + \epsilon/(2k_1k_3)]$. Since the segments L_i and L_j do not intersect for any i and j with $i \ne j$, we have

$$\int_0^\infty f(t)dt \ge \int_0^T f(t)dt \ge \sum_{t_i \le T} \int_{L_i} f(t)dt \ge \frac{\epsilon}{2}\frac{\epsilon}{k_1k_3}M(T) \qquad \text{(A.87)}$$

where $M(T)$ is the number of points t_i not exceeding T. Since $\lim_{T\to\infty} M(T) = \infty$, the integral $\int_0^\infty f(t)dt$ is divergent. This contradicts 2) in (A.85). This contradiction proves the lemma.□

Lemma A.26. *Consider a scalar system*

$$\dot{x} = -cx + p(t) \qquad \text{(A.88)}$$

where $c > 0$ and $p(t)$ is a bounded and uniformly continuous function. If, for any initial time $t_0 \ge 0$ and any initial condition $x(t_0)$, the solution $x(t)$ is bounded and converges to 0 as $t \to \infty$, then

$$\lim_{t\to\infty} p(t) = 0. \qquad \text{(A.89)}$$

Proof of Lemma A.27. See [68]

Lemma A.27. *Consider a first-order differential equation of the form*

$$\dot{x} = -(a(t) + f_1(\xi(t)))x + f_2(\xi(t)) \qquad \text{(A.90)}$$

where f_1 and f_2 are continuous functions, and $\xi : [0,\infty) \to \mathbb{R}^m$ is a time-varying vector-valued signal that exponentially converges to zero and, for all $t \ge t_0 \ge 0$, satisfies

$$|f_i(\xi(t))| \le \gamma_i(\|\xi(t_0)\|)\, e^{-\sigma_i(t-t_0)} \qquad \text{(A.91)}$$

where $\sigma_i > 0$, $i = 1, 2$ and γ_i are class-K functions. If $a(t)$ enjoys the property that there is a constant σ_3 such that

$$\int_{t_1}^{t_2} a(\tau)d\tau \ge \sigma_3(t_1 - t_2),\ \forall t_1 \ge t_2 \ge 0 \qquad \text{(A.92)}$$

then there exists a class-K function γ and a constant $\sigma > 0$ such that

$$|x(t)| \le \gamma(\|(x(t_0), \xi(t_0))\|)\, e^{-\sigma(t-t_0)}. \qquad \text{(A.93)}$$

Proof of Lemma A.27. See [69].

A.7 p times differentiable and smooth bump functions

Differentiable bump functions are widely used in both manufacturing and control fields to smooth out discontinuities between surfaces. However, these functions have been found by non-trial methods so far in the literature, and can only differentiate up to finite times. In manufacturing, smooth transition between discontinuous surfaces in cam profile design currently utilizes various twice differentiable bump functions such as polynomial, harmonic and cycloidal functions, see Chapter 8 in [70]. In this reference, only polynomial bump function can be more than twice differentiable while obtaining harmonic and cycloidal bump functions, which are more than twice differentiable, is a matter of trial and error. In control, one time differentiable bump functions have been used to deal with limited sensing capacity of the agents in coordination control of multiple agents, see [71] and [72], to avoid the use of switching controls [59], which are generally difficult to analyze stability [39]. When the actual dynamics of the mobile agents are taken into account when designing coordination control systems, it is more desired to use bump functions that can be differentiable more than one time. This allows the designer to use the standard backstepping technique [12] to extend the kinematics to dynamics controls. This appendix provides a simple and constructive method to construct smooth and p times differentiable bump functions.

Definition A.28. *A scalar function $h(x,a,b)$ is called a p times differential bump function if it enjoys the following properties*

$$\begin{aligned}
&1)\ h(x,a,b)=1 \quad \textit{if} \quad 0\le x\le a,\\
&2)\ h(x,a,b)=0 \quad \textit{if} \quad x\ge b,\\
&3)\ 0<h(x,a,b)<1 \ \textit{if}\, a<x<b,\\
&4)\ h(x,a,b) \ \textit{is } p \textit{ times differentiable with respect to } x
\end{aligned} \tag{A.94}$$

where p is a positive integer, $x\in\mathbb{R}_+$, and a and b are constants such that $0\le a<b$. Moreover, if $p=\infty$ then the function $h(x,a,b)$ is called a smooth bump function.

Construction of differentiable bump functions is given in the following lemma.

Lemma A.29. *Let the scalar function $h(x,a,b)$ be defined as*

$$h(x,a,b)=1-\frac{\int_a^x f(\tau-a)f(b-\tau)d\tau}{\int_a^b f(\tau-a)f(b-\tau)d\tau} \tag{A.95}$$

with the function $f(y)$ being defined as follows

$$f(y)=0 \ \textit{if}\ y\le 0,\quad f(y)=g(y)\ \textit{if}\ y>0 \tag{A.96}$$

where the function $g(y)$ enjoys the following properties

$$a) \quad g(\tau - a)g(b - \tau) > 0 \quad a < \tau < b, \tag{A.97}$$

$$b) \quad g(y) \text{ is } p \text{ times differentiable with respect to } y, \text{and } \lim_{y\to 0^+} \frac{\partial^k g(y)}{\partial y^k} = 0,$$

$k = 1, 2, ..., p - 1.$

Then the function $h(x, a, b)$ is a p times differentiable bump function.

Proof. We need to verify that the function $h(x, a, b)$ given in (A.95) satisfy all properties defined in (A.94). Property 1) holds because by (A.96) for all $0 \le x \le a$, we have $\int_a^x f(\tau - a)f(b - \tau)d\tau = 0$. Property 2) holds since by (A.96) we have $\int_a^x f(\tau - a)f(b - \tau)d\tau = \int_a^b f(\tau - a)f(b - \tau)d\tau$ for all $x \ge b$. To prove Property 3), we first note from Property a) of the function $g(y)$ given in (A.97) that $\int_a^x f(\tau - a)f(b - \tau)d\tau > 0$ for all $a < x < b$. Therefore, $0 < \frac{\int_a^x f(\tau-a)f(b-\tau)d\tau}{\int_a^b f(\tau-a)f(b-\tau)d\tau} < 1$, which means that Property 3) of the function $h(x, a, b)$ holds. To prove Property 4), we just need to show that $f(y)$ is $p - 1$ times differentiable. We first note that $f(y)$ is p times differentiable except at $y = 0$. Hence, we only need to verify that $f^{(k)}(0) = \left.\frac{\partial^k f(y)}{\partial y^k}\right|_{y=0} = 0$ for any positive integer $k < p$. Clearly, $\lim_{y\to 0^-} f^{(k)}(y) = 0$ since $f(y) = 0,\ \forall y \le 0$. On the other hand, since $f(y) = g(y), y > 0$, from Property b) of the function $g(y)$ we have $\lim_{y\to 0^+} f^{(k)}(y) = \lim_{y\to 0^+} g^{(k)}(y) = 0$, where $g^{(k)} = \frac{\partial^k g(y)}{\partial y^k}$. Since both left- and right-hand limits are equal to 0, we have $f^{(k)}(0) = 0$. Hence Property 4) holds. □

Remark A.30. Several examples of the function $g(y)$ are $g(y) = y^p$, $g(y) = \tanh(y)^p$, $g(y) = \arctan(y^p)$ for any positive integer p, and $g(y) = \sin(y)^p$ for any even positive integer p. It is trivial to check that these functions satisfy all properties listed in (A.97). All the bump functions used in [70] and [71] can be directly derived from Lemma A.29 by picking a proper function $g(y)$.

Corollary A.31. *. If the function $g(y)$ in (A.97) is taken as $g(y) = \exp(-\frac{1}{y})$ then the function $h(x, a, b)$ defined in (A.95) is a smooth bump function.*

Proof. We first note that Property a) of the function $g(y)$ in (A.97) can be proven without a difficulty. We focus on proof of Property b). We note that $g^{(k)}(y) = \frac{\partial^k g(y)}{\partial y^k} = Q_k(\frac{1}{y})e^{-\frac{1}{y}}$ where $Q_k(\frac{1}{y})$ is a polynomial function of $\frac{1}{y}$, and k is any positive integer. We will prove Property b) of the function $g(y)$ in (A.97) by induction. It is clear that $\lim_{y\to 0^+} g^{(1)}(y) = \lim_{y\to 0^+} \frac{g(y)-g(0)}{y-0} = \lim_{y\to 0^+} \frac{e^{-\frac{1}{y}}}{y} = \lim_{\xi\to\infty} \frac{1}{e^{\xi}} = 0$ where $\xi = \frac{1}{y}$ and we have used l'Hopital's

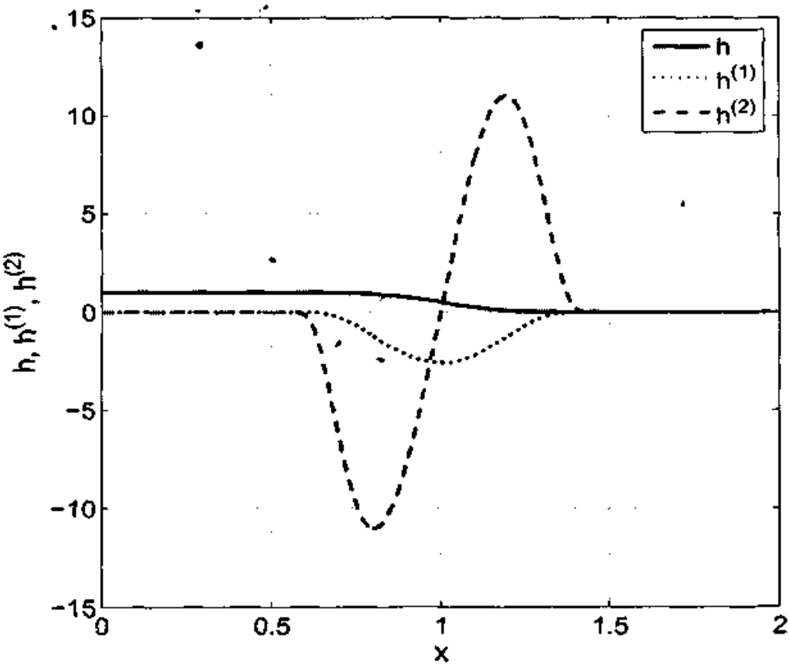


Fig. A.1. *A smooth bump function and its first and second derivatives.*

rule [73]. This means that Property b) of the function $g(y)$ holds for $k = 1$. Assuming that $\lim_{y\to 0^+} g^{(k)}(y) = 0$, we now compute $\lim_{y\to 0^+} g^{(k+1)}(y)$ as follows:

$$\begin{aligned}\lim_{y\to 0^+} g^{(k+1)}(y) &= \lim_{y\to 0^+} \frac{g^{(k)}(y) - g^{(k)}(0)}{y - 0} = \lim_{y\to 0^+} \frac{1}{y} Q_k(\frac{1}{y}) e^{-\frac{1}{y}} \\ &= \lim_{y\to 0^+} \tilde{Q}_k(\frac{1}{y}) e^{-\frac{1}{y}} = \lim_{\xi\to\infty} \frac{\tilde{Q}_k(\xi)}{e^{\xi}} = 0 \end{aligned} \tag{A.98}$$

where l'Hopital's rule has been used, $\xi = \frac{1}{y}$, and $\tilde{Q}_k(\xi) = \xi Q_k(\xi)$ is another polynomial of ξ. Therefore we have proved that $\lim_{y\to 0^+} g^{(k)}(y) = 0$ for any k, which means Property b) of the function $g(y)$ holds for any positive integer p, i.e. p can be equal to infinity. By Definition 1, the function $h(x, a, b)$ is a smooth bump function.□

Fig. A.1 illustrates a smooth bump function and its first and second derivative, with $a = 0.5$ and $b = 1.5$.

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